<u>Proving Mathematical Propositions</u>

A Semiformal Introduction to Formal Mathematics:

How do we know that some mathematical statement is true?

This is THE QUESTION of this course. We want to be able to determine whether a mathematical statement is true or false.

This involves problem solving. By this I mean problem solving in the widest sense: from a simple application of mathematics- like finding the area of a room, to the ultimate in problem solving- creating and writing down a proof. The text is informal. The writing will often appear as if I am talking to you directly, and hopefully (if I am able to anticipate your questions and difficulties) as if we are having a conversation.

Let's start with an example of a problem and informally go through some of the processes and thoughts that could lead to an understanding of the problem and then on to a proof. Suppose someone asks you what happens when we add two even numbers together.

You think about it for a moment,…perhaps try some small examples, 2+ 6 = 8, 2 +2 = 4, 6 + 4 = 10, 18 + 20 = 38, and maybe you wonder what other numbers are even. We probably agree that fractions are neither even nor odd. What about negative numbers or zero? Already we have our first difficulties in problem solving. We need to understand more clearly what question we are asking. For now let's limit the problem to numbers like 2,4,6,8,10,12,14,16,… and so on. We call these the even natural numbers.

After some thought and review of our examples we would probably determine that when we add two even natural numbers the sum is itself even. Now we can write down the proposition that we are thinking about. The sum of two even natural numbers is even.

You may at this point say, "This is obviously true." Now there are two things that could happen here. First, if it is obviously true then we should easily be able to write a proof since it is so obvious. On the other hand, if it isn't really obvious, then we must write a proof because it isn't clear at all that it is even true. Here we have **the** major point of mathematics. Mathematicians say that a proposition is true when there is a proof and not before. No matter how simple or clear an idea is, it is not true in mathematics until a proof is written down and approved by other mathematicians. The purpose of this book is to begin to train you in the skills needed to decide whether something is a proof or not and eventually to teach you to be able to write proofs.

So let us look more carefully at this proposition and begin to think about how we would prove it.

Proposition: The sum of two even natural numbers is even.

Think about the problem: What do we know about even numbers? Not much, it seems. Well, what are some **EXAMPLES**?
Let's see––– 2 is even, so is 4 and 6 and 58,.... What property do they have in common? (Yes, of course, they are even). How can I express this in a more mathematical way? In other words, what does it mean to be even? Hmmmmmm...
Even numbers can always be divided by 2. So every even number must look like (be of the form) 2 times some other number. Do we know what this other number looks like? Hmmmmm?!??
Well, what are some **EXAMPLES**? $8= 2 \cdot 4$, $10= 2 \cdot 5$, $58= 2 \cdot 29$. The other number could be odd, even, prime or composite. It seems like all we have to work with is: an even number is 2 times some other number.

Now what do we want to do with even numbers?
Add two of them. So let's take two of them. They must look like: 2 times <u>some</u> other <u>number</u>. How do we say, " some number" in mathematics?

If it is a natural number (sometimes called the counting numbers, 1,2,3,...) we are talking about, we usually use the letter n and say, "let n be a natural number". If we need another natural number we use "m". So to indicate that we are given two even numbers we write, 2 times n, and 2 times m, where we don't know anything about n and m except that they are natural numbers.

What happens when we add $2 \cdot n$ and $2 \cdot m$?

Well, let's **write it down and try it!!**

(Even though we do not know what will happen, or maybe we don't have a complete plan or idea where to go next, we still **write down** what we do know so that we can see and study it.)

$(2 \cdot n) + (2 \cdot m) = ??$
Wait a minute, isn't that a common factor of 2 in there? Yes, it is.
What if I take that common factor out?
$(2 \cdot n) + (2 \cdot m) = 2 \cdot (n + m)$. How does that help me?

Usually by this time, I've forgotten what I want. So let's look back up at the problem to see what is it we need to find out.
We need to show that the sum is even. How could we do that??
Well, what does it mean to be even? Oh yes, I remember.....,
it means that the sum must be 2 times some number. Hmmm....

Hey, this is Great! $2 \cdot (n + m)$ IS 2 times some number. I'm done! No, not yet. Now I must impress my friends and other mathematicians by writing the proof as if I knew where I was going all along.

Proof:

An even natural number is defined as a natural number which can be written as 2 times a natural number. Let $(2 \cdot n)$ and $(2 \cdot m)$ be any two even natural numbers. Then $(2 \cdot n) + (2 \cdot m) = 2 \cdot (n + m)$ by factoring out the 2. Now since $2 \cdot (n + m)$ is, by definition, an even number, we have shown that:

the sum of two even natural numbers is an even number.

Let us put into practice what we have just learned. For each of the following statements either write proofs to show that they are true or find counterexamples to show that the general statement is false. Make sure that you consider all possible cases.

The sum of two odd natural numbers is even.
The sum of an odd natural number and an even natural number is even.
The product of two even natural numbers is even.
The product of two odd natural numbers is even.
The product of an odd natural number and an even natural number is even

Isn't it interesting how, once we get our mind wrapped around a problem, it continues to think about the problem even when we move on to the next problem. When we immerse ourselves in a problem and give ourselves time to roll it over in the brain, we can often come up with interesting and even useful ways to approach it, and eventually to solve it. But we must first put our time, thought and energy into understanding and trying, no matter how frustrating that might be.

Mathematicians like to expand and generalize on problems. For example, once they figure something out for numbers that can be evenly divided by 2 (even) then they want to look at numbers evenly divisible by 3, and 4, and 5... So what can we say about the product of three consecutive natural numbers? And can we prove it?
What about the sum of three consecutive natural numbers?
Could we prove that the product of any n consecutive natural numbers is evenly divisible by n?

How do we proceed here? How do we know what is acceptable as an argument or proof? What is logical and what isn't?

So, let us begin our informal introduction to formal mathematics. First we study **logic**, where we define and determine what we mean by logic, what constitutes a valid argument and what a proof is.

Our friend said, "I am going swimming today."
That seems like a nice simple statement. We know she is honest and does not want to lie to us; yet, when we meet her for dinner, she tells us that she did not go swimming. Her mother called her and they had a nice long chat. Then when she checked her email she found a rambling note from her brother asking her for advice. So she had to answer that right away. By the time she was done, it was too late to go swimming. Thus, when she said she was "going swimming today", it wasn't true.

Perhaps she could have said, "I am going to go swimming today unless my mother calls or my brother emails or it snows or the water evaporates, or…" Soon we see that we could not list all possible circumstances. Maybe she could say, "I am going to go swimming today unless something stops me from going swimming." But that doesn't seem to be very helpful. In fact it seems almost circular. She might as well say, "I am going to go swimming today unless I don't go swimming today." Now we have a true statement, but it isn't very useful.

Fortunately, we do not often talk like that and, most of the time, we do not need to talk like that. However, in mathematics we DO need to be very specific and careful about what we mean and how we say it.

It is not good enough to say that this rocket ship will take us to the moon when we mean that this ship will take us to the moon if we ignore gravity and the changing mass of the ship. Or that this function describes the state of the economy and will correctly predict how much money will be collected in taxes except that the function does not include the underground economy, bartering for services or complex effects of the behavior of consumers.

In mathematics we must be able to make our statements as precise as possible. We must detail as many of the special cases, assumptions, suppositions and definitions as we can. Otherwise we will not be able to tell whether our statements are true or not.

This is an impossible task in our natural languages such as: English, Japanese, Swahili, Hindi, Spanish, Arabic, ….

Natural languages contain too much ambiguity and too many outright contradictions.

What does "Time flies like an arrow" mean?

Even if we understand that, then what about,

"Fruit flies like a banana." ? [Noam Chomsky]

Words in natural languages have multiple, often contradictory, meanings (flies), and many times the same words are used as different parts of speech (nouns, verbs, adjectives, …). What a word (or a phrase) means often depends upon the context in which it the word (or phrase) occurs. If we say, "The store is full", do we mean that there are a lot of people in the store, or that the store is crowded with merchandise? If we were to say, "The moon is full", then we wouldn't mean either of those things about the moon. Even worse, it is certainly not true that the moon is full everywhere. It may be full from where we stand but from some other place on the earth or from Mars it may not appear full.

In addition, sentences have complex, unclear, sometimes paradoxical content. How would we evaluate the truth of the phrase,

"I can't get no satisfaction." ?

Or even worse, what could this statement mean,

"This statement is false." ???

In English there is actually a phrase for the idea that the same word can mean or stand for two opposite or contradictory things. Such a word is called a Janus word (from the two-faced image of Roman mythology). "Cleave" is a Janus word it means both "to hold onto" and "to split apart". Perhaps even more amazing is the situation where negation does <u>not</u> give the opposite meaning. Inability is the lack of ability; inaction is the lack of action; incomplete means not complete; insensitive means not sensitive; yet inflammable does not mean not flammable, it means the same thing as flammable.

Philosophers and mathematicians from ancient times on down (and more recently computer scientists) have come to the conclusion that we must abandon natural languages if we want to be able to reason truthfully and unambiguously. Therefore to pursue the question

"How do we know when a sentence is true?"

we will restrict the kinds of sentences we will look at.

A <u>proposition</u> (sometimes referred to as a <u>statement</u>) is a sentence that is either true or false.

Questions are not propositions:

Are you there?
Is today a holiday?

Commands are not propositions:

Read carefully.
Stop! ☹
You will be good. ☺

Comments or opinions are generally not propositions:

It is really hot today!
Robins are nice birds.
The moon is a harsh mistress.

Ambiguous or paradoxical sentences are not propositions:

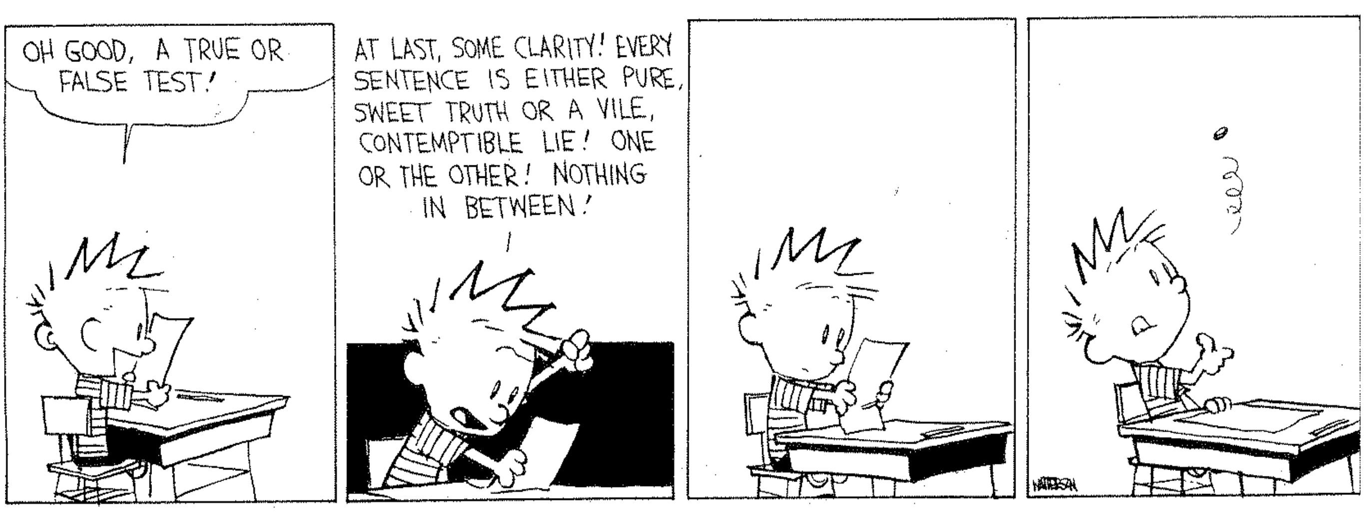
Things fall apart. (some things? all things? most things? apart how?)
This statement might be false.
This statement is false.

Predictions are not propositions:

This piece of paper will be in New York in two years.
I'll bet that you can solve it.
Red is green in another galaxy.

Incomplete thoughts or phrases are not propositions:

If wishes were horses…
The product of two integers…



Propositions are either true or false (but not both). The following are propositions:

1=2. (Of course this is false in our arithmetic)

This paper is in New York now.

If a number is even then its square is even.

If the square of a number is even then the number must be even.

The sum of two odd numbers is an odd number.

Yesterday either Hadley went swimming or she did not go swimming.

There are infinitely many prime numbers.

We said that natural languages had too many problems, and then we proceeded to give all our explanations and examples (except 1=2) in English. We must start somewhere. We must agree on some basic set of symbols, words, or method of communication to get us off the ground. We could actually begin with a set of undefined symbols, a set of rules and a declaration that the only way to use the symbols is defined by the rules. Formal mathematical systems can and do start this way. Once you have more experience with logic and set theory you may want to work your way through such a system. Think of it this way: If you wanted to study the theory of games, it would be better if you had played several different games first before attempting to analyze the structure of the general theory of games. If you had not played any games you would probably find it difficult to understand abstract references to things like turns, playing pieces, or real-time, first person games.

We will start at a less theoretical level and assume we can communicate (carefully of course) in some limited form of English. We will assume we know and understand the meaning of a variety of simple and unambiguous words. We have already defined (given a definition of) what we mean by a proposition.

Do you know what we mean by "define" (or by "to give a definition")?

If not, LOOK it up and **ASK** about it!

In mathematics we try to describe as exactly as possible what we mean by a certain word or symbol. This description is called a definition. We often write the definition in the following form.

Definition 1.1: A proposition (sometimes referred to as a statement) is a sentence which is either true or false.

The word, proposition, which is being defined is underlined. "Definition" is in bold type. Bold type will tell us what part of mathematical speech is being used or described.

This is an informal definition. It uses the word "sentence". It talks about "true" and "false" as if we already know what these things are. This is how we will get started. After some practice you will feel more comfortable with the notion of proposition.

Some people might say that the terms "sentence", "true", and "false" are undefined terms in our system. At this point let us rely on the idea that we do really know what these words mean. We will try to make sure that definitions and results that follow will agree with our notion of what these words mean.

Notice that our definition included the phrase "(sometimes referred to as a statement)". This indicates that the word "statement" can be used interchangeably with the word "proposition". They both mean exactly the same thing. So sometimes we will use the word "proposition" and in the next sentence we will write "statement" to refer to the exact same thing.

Often definitions and other parts of our mathematical system are numbered so that we can keep track of them and refer to them later.

Thus, Definition **1. 1**
$\Uparrow$ $\Uparrow$
means: in chapter 1, this is the first item (definition, proposition,…) described.

At some later stage we might say," see definition 1.1" or "see the definition of proposition".

As examples of propositions we could take sentences like,
"The sea is calm tonight." Or "The product of two prime numbers is a composite number." But this choice of specific sentences is awkward. Many times in mathematics we want to refer to a general object rather than any specific examples. We could say add 8 and 6. The sum is 14, which is an even number. However, this is just one fact. If we wanted to observe that this fact is true for any even numbers we could write- the sum of any two even numbers is an even number. A more concise way to

describe this is to use symbols for the numbers. In algebra we use letters like x and y to stand for any numbers. We might write, “call the number x” or “ let x be a number”. If we need other numbers we usually use y, and then z.

For natural numbers we often use the letters n and m. If we want to indicate that a number is an even natural number, we can write 2n. If we want to indicate that a number is an odd number, we can write 2n+1 or 2n −1 (why does this work?). If the number is prime we can use p and write “let p be a prime number”. If we need to write about a second prime number we often use the letter q. These are symbols that stand for objects. In this case they are letters that stand for numbers. They only mean what we say they mean.

When we want to talk about some objects that are propositions, without referring to any particular propositions, we will use different letters to denote the propositions. Some letters that are often used to stand for or represent propositions are p, q, r, s, t,... or P, Q, R, S, T,.... Sometimes small letters are used, sometimes capitals letters are used.

We will use capital letters, such as P, Q, R, S, T, ... , to stand for or denote propositions.

Questions?? *Do we understand what the word* “denote” means here?

One of our goals is to be able to break down more complicated propositions into simpler statements so that we can better understand what is being asserted in the proposition. For example, what does this mean:

Either the candidate will not run for reelection or it is not the case that the current situation will not be decided without further lack of interference from those who are unaffected by this absence of a detailed denial. (?)

Or this,

A function is continuous at a point c if, for every $\varepsilon > 0$*, there is a* $\delta > 0$ *such that, whenever* $|x - c| < \delta$ *then* $|f(x) - f(c)| < \varepsilon$.

(The definition of continuity at a point)

Or this,

Let f be a function defined on the closed interval [a, b]. If f is continuous on [a, b] and differentiable on (a, b), then there is at least one number c in the open interval (a, b) at which $f'(c) = \frac{f(b) - f(a)}{b - a}$

(The Mean Value Theorem)

We want to be able to take apart longer, more complicated propositions and look at the simpler pieces (that is, the simpler propositions that make up the longer one). If we understand the simpler pieces then we can put them back together and try to understand the bigger statement.

What sorts of words "connect" propositions?
In English, words like "and", "or", "if", "because",,… are used to connect propositions to other propositions. So these words are often called <u>connectives</u>. Let's look at some common connectives.

The word "and" is often used to connect two propositions. For example, the proposition:

The sea is calm tonight and the moon is full,

contains two simpler propositions connected by the word "and".

When we say we want to "understand a proposition" we mean, "when is the proposition true and when is it false." For example, when is the proposition, *The sea is calm tonight and the moon is full*, true? What we are asking is, How does "and" work? Suppose the moon is full tonight but the sea is not calm. In that case, is the proposition *The sea is calm tonight and the moon is full*, true or is it false? Surely most people would say that, in this case, the proposition is false. Remember, by definition, a proposition must be either true or false.

In English, when we use "and" to connect 2 propositions, we mean that the entire proposition is true only when <u>both</u> of the simpler propositions are true. If one of the propositions or the other or both of them are false then we say that the combined proposition is false. Does that make sense?

This is something of a mess to explain in words each time, so in mathematics we use abstraction and notation to focus on the important aspects of the ideas. Here, the important facts are whether the statements are true or false. Since a statement is either true or false, each statement has the possibility of 2 values, either true or false.

We use notation to express this abstractly. Pretend we have a proposition. "What proposition?", you might ask. But it doesn't matter. We often say, "let x be a number", and we don't have a specific number in mind. In fact, x might stand for lots of numbers, for example, x>6. So, when we say, "pretend we have a proposition" we might as well say "let P be a proposition" where we are using the letter P as a variable. In general, we use these letters as variables which stand for propositions. We call them propositional variables.

So, let P be a proposition. To show the possible truth values (either True or False) of P, we use a <u>truth table</u>. And because we don't want to write True each time, we abbreviate True with the single letter

T. Similarly we write F to represent the possibility that the proposition is False. We write,

P
T
F

This truth table shows that there are 2 possible truth values for the proposition P. Do you understand what is meant by a "truth value"? This is just an elaborate way of saying that a proposition can be either true or false. This is like the idea that a variable, x, can have different values; it could be any one of many numbers. Our proposition variables, P, Q , R,.. have only two possibilities. Each one is either true or false.

Suppose that we started with two statements, P, Q. How many different ways could we express the truth values of these two statements?

Well, both of them could be true. Both of them could be false. P could be true and Q could be false. P could be false and Q could be true. Are there any other possible combinations of truth values for P and Q? (….) Pause and take some time to think about the previous ideas and question. Try to answer it.

Let's put this information into a truth table. Above we listed the 4 possible truth values for two statements. So we know that if we have 2 different letters then there are four possible truth values for the 2 letters, so there will be 4 rows in the truth table. We want to list those truth values in a systematic way and the same way each time so that we can easily read and compare truth tables.

Starting with the right-most letter, in this case Q, we put truth values in under that letter, alternating T and F down the column.

P	Q
	T
	F
	T
	F

Then we move over one column to the left (in this case we move over to the column headed by P) and alternate T and F by twos, first two T's then two F's.

	P	Q	
Two Trues ⇒	T	T	Alternating True
⇒	T	F	then False
Two Falses ⇒	F	T	then True
⇒	F	F	then False

Now we go back to our statement which used "and" as a connective, so that we can answer the question, when is the statement,

The sea is calm tonight and the moon is full, true?

The answer can be nicely demonstrated with a truth table. We will let P stand for the statement *The sea is calm tonight*. Let Q stand for the statement *the moon is full*. Remember that our understanding of how "and" works is that an "and" statement is true only when both of the simpler statements are true. Otherwise, the "and' statement is false.

Here is the truth table:

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

Thus, the statement *The sea is calm tonight and the moon is full*, is true only when both the statement, *The sea is calm tonight*, is true and the statement, *the moon is full*, is true.

Instead of writing "and" each time, logicians (people who study logic) use notation to abbreviate "and". The most common symbol used to stand for "and" is $\wedge$ (the letter v upside down). Thus we write P and Q, as $P \wedge Q$. And our truth table becomes:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

You may have noticed that this recent discussion has all been carried out in English and it assumes you understand many different words and concepts which have not been defined. If this were a more formal description of "and", we might write:

Definition 1.2: For propositions P and Q, $P \wedge Q$, is defined to be T when both P and Q are T, and is defined to be F otherwise. The symbol, $\wedge$, can be read as "and".

The fancy name for the "and " connective is conjunction. So we say the proposition: 17 is a prime and 19 is a prime, is a conjunction. Any "and" statement is a conjuction.

Using this as a definition we know when $P \wedge Q$ is T, and when it is F, based on whether P is T or F and whether Q is T or F. We do not have to give deeper meaning (or in fact any meaning) to these letters, P,Q,T, and F. We could just play a formal game using P's, Q's, T's and F's without assigning any meaning to these objects ("formal" in this sense meaning, based on the form of the statement $P \wedge Q$ rather than on the specific meanings of P, Q, T or F). This can be useful in finding the essence of the argument without looking too closely at the individual statements. However our long term goal is to help us understand how to reason clearly, so we will assume that P and Q are important and interesting statements, that T means True, that F means False, and that "$\wedge$" means "and" (at least in the limited way we use "and" in our logic, which is some part of the way we use and understand "and" in the natural language, English).

Another English word which acts as a connective between propositions is "or". In English we use the word "or" in several different ways. In our logic we use the word "or" in what is called the inclusive sense. We define P or Q to be T, if either P is T , or Q is T or both P and Q are T. This last case is the inclusive case- we include the use of "or" when both P and Q are T. For example, we might say: We are looking for a number that is prime or ends in 3. This means it could be a prime (e.g. 7) OR it could end in 3 (e.g. 63) OR it could be both a prime and end in 3 (e.g. 23). The symbol we use for "or" is $\vee$. So we write $P \vee Q$ and read it as "P or Q". This is sometimes called the vel operator. The symbol looks like the letter v. Vel is the Latin word for "or". The fancy name for "or" is disjunction. So the proposition: either n is even or n^2 is even, is a disjunction.
The truth table for "or" is

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.3: For propositions P and Q, $P \vee Q$, is defined to be F when both P and Q are F, and is defined to be T otherwise. The symbol, $\vee$, can be read as "or".

In English, we also use "or' in an exclusive sense. We make statements like, "You may have this or you may have that", and we clearly mean that you may have one or the other <u>but not both</u> (exclusive). In fact, we used "or" in exactly this way when we wrote that a proposition is either true or false. By this we meant that it could be one or the other but not both. This is not what we mean by "or" in our logic. The symbol, $\vee$, stands for the inclusive or and means one or the other or both.

e.g. When we write

"Some numbers are a multiple of 3 or a multiple of 5",

we mean they could be a multiple of 3 OR a multiple of 5 OR

a multiple of both 3 and 5.

Question: So, how does "or" work?

Often we want to describe the opposite or negative of a proposition. Instead of writing: p is a prime, we may want to write: p is not a prime. This second proposition is the negation of the first one.

Definition 1.4: The <u>negation</u> of a proposition P, denoted by $\neg P$, is defined to be F when P is T and is defined to T when P is F. The symbol, $\neg$, can be read as "not".

The truth table for negation is

P	$\neg P$
T	F
F	T

There are at least two other notations commonly used for negation. Some times the tilde, ~, is used to represent negation. Instead of $\neg P$ we might see ~ P.

A line over the propositional letter, $\overline{P}$, is also often used as a notation for negation. We will see that this notation is also used to represent the complement of a set, which is the set of elements that are NOT in the set. And we will see that there is a connection between these ideas. That is why the overline notation, $\overline{P}$, is sometimes used for negation.

We often want to find the negation of a proposition. If it is a simple statement like: "Today is Tuesday", then it is easy to write the negation in English: "Today is not Tuesday".

We could also just put the words: “It is not the case that…” in front of any statement we wish to negate although this is often not very helpful in trying to understand the meaning of the statement. Would you want to see “It is not the case that $x<12$” or would you rather see “$x \geq 12$”?

Two other connectives are implication (also called conditional), denoted by $\rightarrow$ and biconditional, denoted by $\leftrightarrow$. We **define** these connectives by their truth tables:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

This is the same as $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

We need to discuss implication (denoted by $\rightarrow$) in some detail because it is so important to mathematics and because in many ways it seems to be very different from and ($\wedge$), or ($\vee$) and negation ($\neg$) .

$P \rightarrow Q$ may be read as ”P implies Q” or as “If P then Q”.
In the implication, $P \rightarrow Q$, P is the hypothesis and Q is the conclusion.
In the implication, $S \rightarrow R$, S is the hypothesis and R is the conclusion.
The only time an implication is False is when the first part (the hypothesis) is True and the second part (the conclusion) is False. In general we say, True implies False is False. This is a **very important** case. It will be used repeatedly when we consider arguments. Most people would agree that the statement, If P then Q, should be considered False if P were True but Q were False. That is, True implies False certainly does not sound like a reasonable way to fashion a discussion. Also, most people would agree that the statement, If P then Q, should be considered True when both P and Q are True. After all, True statements should lead to True statements.

In the other two cases, the hypothesis is False. This gives rise to some odd ideas, so let us think about them for a moment. What could it mean if our hypothesis is False? That is, what if we start with a false statement?

Mathematical systems of thought are powerful and useful because, among other reasons, they are consistent. This means that if we start with True statements and use only logically correct reasoning we cannot prove False statements. But what if we start with False statements? Perhaps you have seen one of those “proofs” that begin with (or somewhere include) a False statement and then proceed to “prove” some other false statement. This is the other side of consistency. According to the logical foundations of mathematical systems, in an inconsistent system, any false statement can be proved. If you start with $1+1= 3$,

then you can prove that 2+2= 6, or that 5+7= 108. Thus if we start with a false statement in an implication then the entire implication can be shown to be true even if it seems to make no sense. i.e. <u>if</u> 1+1= 3 <u>then</u> 5+7=108. That is exactly how we use this part of implication. Sometimes even in a natural language like English we use implications this way. For example, when I say, "If they win, then I'll eat my hat". The idea, of course, is that I really believe they have no chance to win. Thus the statement "they win" is thought to be blatantly False. Yet I intend the hearers of my statement to take me seriously and accept my entire statement as a true statement. Here is another illustration:
If $-3 = 7$ then $-3 + 1 = 8$.
Clearly $-3 = 7$ is false.
Yet we are reluctant to say that the entire statement is false because
<u>IF</u> $-3 = 7$, then of course it would be true that $-3 + 1 = 8$.

Even if this notion that <u>False implies anything is True</u> is not satisfying to you, nonetheless our **definition** tells us exactly how we are defining this connective and that is how we will use it.

Go back and review the truth table for implication. Make sure that you know it well. One way to remember it is:
$P \rightarrow Q$ is always true, except when P is True and Q is False.

More formally we define implication this way:
Definition 1.5: For statements P and Q, $P \rightarrow Q$, is defined to be F when P is T and Q is F, and is defined to be T otherwise.
The symbol, $\rightarrow$, can be read as "implication" or "conditional". $P \rightarrow Q$ can be read as "P implies Q" or "If P then Q".
$P \rightarrow Q$ can also be read as "P is <u>sufficient for</u> Q" and as "Q is <u>necessary for</u> P".

Now that we know a little bit of logic we can begin to see how to use it. Many mathematical statements are implications:

<u>If</u>...... <u>then</u>......

Remember our first example of a proof: When we add 2 even natural numbers we get an even number.

We can rewrite this as an implication. That is, we can write it in the form: <u>If</u>...... <u>then</u>......
Let's try to do this. What are we assuming here? What do we start with? What is our hypothesis? Well..., we are trying to add 2 natural numbers. So that is our assumption (hypothesis).
Or you may think of this as what we are given. We could say: <u>If</u> we are given two even numbers... .

We are supposed to conclude or observe something about these 2 numbers, namely that their sum is an even number. So we can write the conclusion as: then the sum of those 2 numbers is even.

Thus we now have:

If we are given two even numbers then the sum of those 2 numbers is even.

This means the same as our original statement and it is now written as an implication.

When is this implication true? How could this implication be false? If you do not know the answer to these questions, go back and reread the section on implications, study the truth tables and ask yourself if we could get a false in the truth table for this implication.

An implication, $P \rightarrow Q$ (*ANY* implication, $P \rightarrow Q$) is false when P is true and Q is false. In all other cases the implication is true. That is, if P is true and Q is true then the implication is true. Even in the dumb cases, when P is false, we declare that the whole implication is true (no matter whether Q is true or false).

Therefore to answer the question: "How do we prove that an implication is true?", all we need to do is show that when the first part of the implication, the hypothesis, is true, then the second part of the implication, the conclusion, is also true. The way this works in practice is that we assume the first part to be true. In other words, when we say "if we are given 2 even numbers", we are assuming that we have 2 even numbers and not 2 odd numbers, or an odd number and a fraction,…. This may seem a bit silly but it is very important for our later work on more complicated problems.

The point is that to prove an implication we assume that the first part of the implication is given to us (that it is true, that we have whatever the first part describes) and our goal is to prove that the second part of the implication is true. We will need to use the information assumed in the first part to carry out the proof. If we can prove the second part without using the first part then it doesn't need to be an implication.

An "implication" is sometimes called a "conditional".

Thus, the "biconditional", $P \leftrightarrow Q$, means implication in both directions: $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

Definition 1.6: For statements P and Q, $P \leftrightarrow Q$, is defined to be T when P and Q have the same truth value (that is, when both P and Q are true, or when both P and Q are false), and is defined to be F when P and Q have the opposite truth values (that is, when P is true and Q is false, or when P is false and Q is true). $P \leftrightarrow Q$, can be read as "P if and only if Q", or "P implies Q and Q implies P".

Notation is a very important part of mathematics and you should make it part of your writing style. “If and only if” is often abbreviated as “**iff**”.

Example: A triangle is a right triangle iff (if and only if) the sum of the squares of the two shorter sides equals the square of the longest side.

This proposition means two separate and different statements.

One is the well known Pythagorean Theorem: If we have a right triangle then the square of the longest side (called the “hypotenuse”) is equal to the sum of the squares of the other two sides.

The other says something different. It says that if the sum of the squares of the two shorter sides equals the square of the longest side. then the triangle must be a right triangle.

In the first statement we start with a right triangle. That fact forces some relationship about the lengths of the sides of the triangle.
In the second statement, it is the relationship of the lengths of the sides that forces the triangle to be a right triangle.

Compound Propositions

When a statement consists of more than one simple statement we call it a compound statement or compound proposition. Both of the following statements are compound propositions.
$P \vee (\neg Q)$ includes the simple proposition P and the simple proposition Q.

$(S \rightarrow \neg P) \wedge (Q \vee R)$ includes 4 simple statements.

To construct a truth table for a compound proposition, first we count how many different letters (statements) there are in the compound statement. For example (i) $P \vee (\neg Q)$ has 2 different letters (statements), (ii) $(P \wedge Q) \wedge \neg P$ has 2 different letters (statements) and (iii) $(\neg P \wedge R) \vee (R \rightarrow Q)$ has 3 different letters (statements).

Next we list the letters across the top of the table.
Example: $P \vee (\neg Q)$ has two letters so we list them across the top of a truth table.

P	Q

We already know that if we have 2 different letters then there are four possible truth values for the 2 letters, so there will be 4 rows in the truth table and we know the pattern of the entries. We start with the right most letter (Q) and we alternate T and F down that column.

Then we move over one column to the left (in this case we move over to the column headed by P) and alternate T and F by twos, first two T's then two F's.

	P	Q
Two Trues ⇒	T	T
⇒	T	F
Two Falses ⇒	F	T
⇒	F	F

Now we add more columns as we need them and fill them in. Q is negated in the proposition so we add a column for that and fill it in. Remember how negation works.

P	Q	$\neg Q$
T	T	F
T	F	T
F	T	F
F	F	T

Next we add columns by putting down propositions that will build up to the proposition with which we started. In this case we already have $\neg Q$ so we only need to add the final proposition, $P \vee (\neg Q)$.

P	Q	$\neg Q$	$P \vee (\neg Q)$
T	T	F	
T	F	T	
F	T	F	
F	F	T	

Finally we fill in the last column(s) using the rules we know for our connectives. In this case, remember how "or" works and fill in the last column.

P	Q	$\neg Q$	$P \vee (\neg Q)$	
T	T	F	T	(because P is True)
T	F	T	T	(use either P is True, or $\neg Q$ is true)
F	T	F	F	(both P and $\neg Q$ are False)
F	F	T	T	(because $\neg Q$ is True)

Let's construct a truth table for a proposition that has three different letters: $(\neg P \wedge R) \vee (R \rightarrow Q)$. First we observe that, since there are three different letters, P, Q, and R, we must have 8 rows in the truth table.

Stop for a moment and ask yourself why this should be true. Try to write down all the possible truth values for three propositions, P, Q and R. Hmmmm… well, all three could be true. That would be one row of the truth table. P could be true, Q could be false and R could be false. This is another row in the truth table. How many possible ways are there to make each letter either T of F? Write them all out now.
If you didn't list 8 different ways then look carefully at the next two truth tables and try to find the ones you missed.

We want to list all the possible ways and we want to list them systematically to be sure that we never leave any out of our truth table and also so that it is easy to compare truth tables.

Here is a method for listing the 8 rows in a truth table for a compound proposition that has three different simple statements. We put P, Q, and R on the top of the table and alternate T's and F's under the right-most letter R (for eight rows). Then under Q we alternate by twos- two T's then two F's, then two T's, then two F's (for eight rows).

	P	Q	R	
		T	T	
		T	F	⇐ Alternate T and F
Alternate 2T's		F	T	
and 2F's ⇒		F	F	
		T	T	
		T	F	
		F	T	
		F	F	

Finally, in the column under P we alternate by four at a time, first four T's then four F's.

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

This is the standard truth table for any statement that has three different letters in it. We know that, with three different letters for statements, there will be exactly eight rows. The first column has four T's then four F's. The second column has two T's then two F's then two T's then two F's. The third column alternates T and F for the eight rows.

Now we add a column for $\neg P$ (since $\neg P$ occurs in the proposition $(\neg P \wedge R) \vee (R \rightarrow Q)$). It is often convenient to add a negation column next to the letter which is being negated, if that is possible. In this example we can (and will) add $\neg P$ in the column just to the left of the column for P. If we needed to add $\neg R$, we could add it to the right of the column with R. Of course, if we needed $\neg Q$ we might not be able to put it right next to the column for Q.

$\neg P$	P	Q	R
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F

Now we want to build up the parts of the proposition. We see that we will need to find the truth values for $\neg P \wedge R$ and the truth values for $R \rightarrow Q$, so that we can "or" ($\vee$) them together. Thus we add columns for these two pieces of the proposition and one last column for the proposition itself.

$\neg P$	P	Q	R	$\neg P \wedge R$	$R \rightarrow Q$	$(\neg P \wedge R) \vee (R \rightarrow Q)$
F	T	T	T			
F	T	T	F			
F	T	F	T			
F	T	F	F			
T	F	T	T			
T	F	T	F			
T	F	F	T			
T	F	F	F			

Finally we fill in the truth values for the pieces that make up the entire proposition and for the entire proposition itself, using our rules for the connectives. For example, $\neg P \wedge R$ will only be True when both $\neg P$ is True (that is, when P is False) and R is True. For the last column, we "or" together the piece $\neg P \wedge R$ with $R \rightarrow Q$. So the proposition $(\neg P \wedge R) \vee (R \rightarrow Q)$ will be True when either $\neg P \wedge R$ is True or $R \rightarrow Q$ is True or both are True.

Here is the complete truth table for the proposition $(\neg P \wedge R) \vee (R \rightarrow Q)$.

$\neg P$	P	Q	R	$\neg P \wedge R$	$R \rightarrow Q$	$(\neg P \wedge R) \vee (R \rightarrow Q)$
F	T	T	T	F	T	T
F	T	T	F	F	T	T
F	T	F	T	F	F	F
F	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	T	F	F	T	T
T	F	F	T	T	F	T
T	F	F	F	F	T	T

This truth table tells us when the statement $(\neg P \wedge R) \vee (R \rightarrow Q)$ is true and when it is false based on the truth or falsity of the statements P, Q and R which make up the larger statement.

Later on we will prove this statement:

Proposition 1.7: Let n be the number of different (unique) letters in a compound proposition then the number of rows in the truth table is 2^n.

For now let us just observe that this proposition is true for our small examples

(i) $P \vee (\neg Q)$ has 2 different letters, so its truth table will have $2^2 = 4$ rows.

(ii) $(P \wedge Q) \wedge \neg P$ has 2 different letters, its truth table will have $2^2 = 4$ rows.

(iii) $(\neg P \wedge R) \vee (R \rightarrow Q)$ has 3 different letters, its truth table will have $2^3 = 8$ rows.

(iv) $(S \rightarrow \neg P) \wedge (Q \text{ v } R)$ has 4 different letters, its truth table will have $2^4 = 16$ rows.

(Here each letter represents a simple proposition.)

As we discussed earlier, even though the proposition listed above seems to be true, we MUST prove it for it to be a valid part of mathematics.

We would like to know when two statements say the same thing, that is, when the logical information in one statement is the same as the logical information in the other. The simplest case of this is double negation. The statements, “This is my desk” and “It is not true that this is not my desk” contain the same information. The second statement has two negatives in it and in some sense they cancel each other out. We want to make this idea exact. So we say:

Definition 1.8: Two propositions are equivalent if they have the same truth values for all possible assignments of T and F in their truth tables.

Observe that this definition uses truth tables to decide when two propositions are equivalent. This is somewhat formal in the sense that we do not appeal to our understanding of what the statements say or mean; we merely check to see that both statements have the exact same truth value in each row of the truth table.

Example: Consider the two propositions P and $\neg\neg P$. Let’s look at the truth table. We construct a table with two rows since there is only one letter being used. We have a column for P and we have a column for $\neg P$ since we need $\neg P$ to build up to $\neg\neg P$.

P	$\neg P$
T	F
F	T

Finally we add a column for $\neg\neg P$ and we fill in the truth values.

P	$\neg P$	$\neg\neg P$
T	F	T
F	T	F

Note that P and $\neg\neg P$ have exactly the same truth values in each row. This means that they are equivalent.

We might also observe that P and $\neg P$ do not have the same truth values in some rows (in fact, they do not have the same truth value in any row). The fact that these two statements differ in at least one row allows us to say that they are not equivalent. None of this should be any surprise to anyone. In ordinary English we see quite clearly that a statement and its negation are not equivalent in meaning and we usually agree that a statement and its double negation are equivalent.

We are interested in finding out which propositions are equivalent to which other propositions.

Notation: The symbol $\equiv$ will be used to denote the fact that two propositions are equivalent.

For example, we write $P \equiv \neg\neg P$ to indicate that P and $\neg\neg P$ are equivalent and we call this particular equivalence <u>double negation</u>.

Consider the statement: 7 is a prime number. Call this statement P.
The negation of P ($\neg P$) is: 7 is not a prime number.
And if we negate again ($\neg\neg P$), we could write: It is not the case that 7 is not a prime number. This is awkward but we can see that $\neg\neg P$ and P contain the same information.
Is it possible that a double positive is ever the same as a negative?
Yeah, right ! ?!

Let's look at a statement involving "and".
Example: The sea is calm and the moon is full.
What is the negation of this statement?

Here are some famous examples of equivalent propositions. These are named after the logician Augustus DeMorgan.

DeMorgan's Laws

(i) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

(ii) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

Equivalence (i) says, the negation of an "or" proposition is the negation of the first statement "and"ed with the negation of the second statement.

In other words, to negate a proposition that consists of two statements connected by an "or", we flip the "or" sign to an "and" sign then we negate each of the individual statements.

Equivalence (ii) says, the negation of an "and" proposition is the negation of the first statement "or"ed with the negation of the second statement.

In other words, to negate a proposition that consists of two statements connected by an "and", we flip the "and" sign to an "or" sign and negate each of the individual statements.

Check that the above equivalences, DeMorgan's Laws, are correct by constructing truth tables for them. Verify that the statements in equivalence (i) have the same truth values in each row. Do the same thing for equivalence (ii). Write out the truth tables now, before reading on.

You should **collect equivalences and write them down in a list.**

Now let's look at some examples in English.
Practice!: <u>Write the negation</u> of each of these statements.
"The tree is tall and the forest is not large."
"The sun is hot yet the earth is not warm."
"The color is not red or the light is dim."
"They take 5 courses and do not live on campus."
"Either the computer is broken or the price is not right."

DeMorgan's Laws tell us how to negate propositions that have an "and " in them and how to negate propositions that have an "or" in them.

One of DeMorgan's Laws tell us how to negate "or" statements. For example, the negation of the proposition: $x<3$ or $x>8$ is $\neg(x<3 \text{ or } x>8)$ which, by one of DeMorgan's Laws, is equivalent to $(x \not< 3 \text{ and } x \not> 8)$. We could write this as

$\neg(x<3 \text{ or } x>8) \equiv (x \not< 3 \text{ and } x \not> 8) \equiv (x \geq 3 \text{ and } x \leq 8)$.

This last conjunction we read as: x is greater than or equal to three AND x is less than or equal to 8. We can write this as $3 \leq x \leq 8$.

Note that this is an "and" statement.

Similarly, the proposition $-2 \leq x < 7$ is an "and" statement $-2 \leq x$ <u>and</u> $x < 7$, so it can be negated by using DeMorgan's Law for "and", $\neg(-2 \leq x \text{ and } x < 7) \equiv$

$(\neg(-2 \leq x) \text{ or } \neg(x < 7)) \equiv (-2 \not\leq x \text{ or } x \not< 7) \equiv (x<-2 \text{ or } x \geq 7)$.

Thus it is relatively easy to negate simple propositions and propositions involving "and" or "or" connectives.

But what is the negation of an implication? For example, write down the negation of the implication: If the sun is shining then we go swimming.

→ Try this *right now*. Write a proposition that is the negation of the proposition: If the sun is shining then we go swimming.

Now test your answer by comparing the truth tables of the proposition and your negation of the proposition. Are they exactly the opposite (wherever the proposition is true is your negation false and where your negation is true is the proposition false)?

It turns out that many humans have some difficulty with negating implications. Fortunately there is an easy way to negate propositions. All we need is an interesting little equivalence.

Proposition 1.9 : $P \to Q \equiv \neg P \vee Q$
Proof: Construct the truth table for these two propositions and see that they have the same truth value in each and every row. (Do this now.)

Go back and look at the definition of equivalent. So far this is the only way we have to check whether two propositions are equivalent or not. That is why when we wrote the proof above we said "construct the truth table for these two propositions…". We were using the definition of equivalent to prove that the two statements were equivalent.

This is an important equivalence because it provides a bridge between statements with an implication and statements with an "and" or an "or". Up to this time we studied implication independently and we had no way to move back and forth from implication to conjunction or disjunction. Now, however we can turn a proposition with an implication in it into a proposition with an "or" and vice versa.

Remember our example:
If the sun is shining then we go swimming.

Let P be the proposition: the sun is shining
and let Q be the proposition: we go swimming.
Then our proposition in symbolic terms is the implication**:** $P \to Q$.
By **Proposition 1.9:** $P \to Q \equiv \neg P \vee Q$.
Thus, If the sun is shining then we go swimming, is equivalent to
Either the sun is not shining or we go swimming.

When we say they are equivalent we mean that their truth tables would be the same. Where one is true, the other is true and similarly for false. Note that the only way that $P \to Q$ (If the sun is shining then we go swimming) could be false is if P is true (the sun is shining) and Q is false (we don't go swimming). And the only time $\neg P \vee Q$ (Either the sun is not shining or we go swimming) could be false is if both $\neg P$ and Q are false.
That is, if $\neg P$ is false, so P is true (the sun is shining) and Q is false (we don't go swimming). Thus, $\neg P \vee Q$ contains the same logical information as $P \to Q$. They are logically equivalent.

If we stop and think about what we might intend by the statement:
If the sun is shining then we go swimming, we might see that

all we are saying is that, if the sun is shining something will happen. We don't care about the situation if the sun doesn't shine.

Another Example: Quadratic formula

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

What does this statement say?

Can you write this as an "or" statement? Does this make sense to you?

Finally we are ready to determine the negation of an implication

Proposition 1.10: $\neg(P \to Q) \equiv \neg(\neg P \vee Q) \equiv \neg\neg P \wedge \neg Q \equiv P \wedge \neg Q$

Proof: $\neg(P \to Q) \underset{\substack{\uparrow \\ \text{Proposition 1.9}}}{\equiv} \neg(\neg P \vee Q) \underset{\substack{\uparrow \\ \text{DeMorgan's Law}}}{\equiv} \neg\neg P \wedge \neg Q \underset{\substack{\uparrow \\ \text{Double Negation}}}{\equiv} P \wedge \neg Q$

Notice that we justified EVERY equivalence in the proof.

Thus the negation of an implication is not an implication. It is an "and" statement. If we think about it for a bit, it does make sense.

What is the negation of the implication in the Quadratic formula?

Note that since the Quadratic formula statement is true, its negation must be false.

Can you write the negation of the following statement?

If the moon is full then the sea is calm.

Try it *right now*. Write a proposition that is the negation of the proposition above.

The negation of "IF the moon is full THEN the sea is calm"
is "The moon IS full and (yet) the sea is not calm".
You should be able to justify each step (each equivalence) in a sequence of equivalences as is done for Proposition 1.10. If any justification is not clear, ask about it. One of our goals is to learn how to justify a sequence of statements that lead to a conclusion. That is, we want to learn how to prove statements.

There are statements that are related to an implication, $P \to Q$. These are: the converse, $Q \to P$, which switches the hypothesis and the conclusion of the original implication, the inverse, $\neg P \to \neg Q$, which leaves the hypothesis and conclusion in the same places as the original implication but negates the hypothesis and negates the conclusion, and the contrapositive, $\neg Q \to \neg P$, which switches the hypothesis and conclusion and also negates them (we could remember this as "switches *and* negates" or "does both things").

Definition 1.11:

The <u>converse</u> of the implication, $P \to Q$, is defined to be the proposition $Q \to P$.
The <u>inverse</u> of the implication, $P \to Q$, is defined to be the proposition $\neg P \to \neg Q$.
The <u>contrapositive</u> of the implication, $P \to Q$, is defined to be the proposition $\neg Q \to \neg P$.

If we construct truth tables for these propositions we will discover something interesting.

$\neg P$	P	Q	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$	$Q \to P$	$\neg P \to \neg Q$
F	T	T	F	T	T	T	T
F	T	F	T	F	F	T	T
T	F	T	F	T	T	F	F
T	F	F	T	T	T	T	T

In each row the truth value for $P \to Q$ is the same as the truth value for $\neg Q \to \neg P$. This says that the two propositions are equivalent. So we can write $P \to Q \equiv \neg Q \to \neg P$.
Or, in other words, an implication is equivalent to its contrapositive.

Examples:

The statement: If m^2 is an even number then m is an even number, is logically equivalent to the statement:

If m is not an even number then m^2 is not an even number.

[And, in this case, the second statement (the contrapositive) is easier to prove!]

Let P be the statement: "m^2 is an even number" and let Q be the statement: "m is an even number". So, "If m^2 is an even number then m is an even number" can be written symbolically as $P \to Q$. And, "If m is not an even number then m^2 is not an even number" is the contrapositive $\neg Q \to \neg P$. (Do you see this? If not, ask a question about it.)

The statement: If a function is differentiable then it is continuous, is logically equivalent to the statement:

If a function is not continuous then it is not differentiable.

Similarly, the converse and the inverse are equivalent, $Q \to P \equiv \neg P \to \neg Q$. (The inverse is the contrapositive of the converse!.)

Note however that the statement

$(P \to Q)$ is <u>not</u> equivalent to its converse $(Q \to P)$.

As an example consider the statement:

If angle A and angle B are both right angles then the size of angle A is equal to the size of angle B.

This is a true statement. (isn't it?)

What is the converse of this statement?

If the size of angle A is equal to the size of angle B then angle A and angle B are both right angles.

And this is clearly false.

So a statement can be true while its converse is false

Here is another example:

If a function is differentiable then it is continuous, (this is a true statement); however, its converse: if a function is continuous then it is differentiable, is not true in general.

The function $f(x) = \begin{cases} x^3, & \text{if } x \le 2 \\ -3x+14, & \text{if } x > 2 \end{cases}$ is continuous at x =2, but is not differentiable at x =2.

Consider the one-sided limits of $f(x)$, as x approaches 2 from above and from below. Then consider the one-sided derivatives of $f(x)$, as x approaches 2 from above and from below.

Example:

The sum of two even numbers is an even number.

Is this a true statement? What kind of statement is it?

As we observed before this can be written as an implication:

If two numbers are even then their sum is even.

Since it is an implication, we can ask, "what is the hypothesis?"...

"two numbers are even"

Can we write this in a way that tells us something or so that it is easier to see what this statement gives us?

"two numbers are even" --- Does this mean that every two numbers are even? I don't think so. I don't think that is what is meant here. So what do we mean by "two numbers are even"?

We might say, “suppose we have two even numbers” or “given two numbers both of which are even” or “let a and b be two even numbers” or “let 2n and 2m be two even numbers”. In other words we are assuming that the hypothesis is a true statement, and thus we have two even numbers. We don’t know what numbers they are, in fact we know very little about these numbers. The only thing we know is that they are even. Another way to say this is “take two even numbers” or “suppose that we have two even numbers”. This is our hypothesis and we can give this statement a name, call it P.
Now, what is the conclusion of the implication? What do want to prove?
We want to prove that the sum of these two numbers is an even number.
Let’s call this statement Q.
Can you prove that $P \rightarrow Q$? (Write out a simple proof now.)

Now let’s look at the converse of this statement.
Write out the converse in symbols? ($Q \rightarrow P$)

Let’s write out what these symbols mean.
Q means that ”the sum of two numbers is even”.
Since Q is the hypothesis in $Q \rightarrow P$, then Q is what we get to assume.
So assume that the sum of two numbers is even.

Now what do we want to prove? We want to prove P based on Q.
What does P say? P says that “the two numbers must be even”.
Is this true? Can we prove it?

The answer is NO. It is not true. Just because the sum of two numbers is even does NOT mean that the numbers we started with must be even. How do we show this? All we need to do here is give an example where the hypothesis is true and the conclusion is false (remember our definition of what it means for an implication to be false).
I am going to go slow here to try to carefully write out the thinking.
We need to construct an example using actual numbers.
We want the sum to be even but we want to try and make the individual numbers not both even. So let’s start there. Perhaps the two numbers could be 4 and 7.
Certainly they are not both even. However, the sum of the two numbers is odd, not even, so this will not satisfy the hypothesis. Okay, let’s try some other numbers, say, 3 and 5.
3+5 = 8. 8 is even, but 3 and 5 are NOT both even. Thus we have shown that the statement, $Q \rightarrow P$, which is
“If the sum of two numbers is even then the two numbers are even”
is NOT TRUE. And we have shown this general statement be false by providing a <u>counterexample.</u>

We have now shown that an implication, $P \rightarrow Q$, can be true while its converse, $Q \rightarrow P$, is false. Therefore, an implication and its converse do not say the same thing. This is very important for us since so many mathematical statements are implications.

It certainly can happen that a statement and its converse are both true. The Pythagorean Theorem is an example of this.
The important thing to remember is that the truth or falsity of a statement is independent of the truth or falsity of its converse. Both could be true; both could be false or one could be true and the other false.

Sometimes people use the contrapositive in a way to mislead us.
Example: If it isn't fresh, it isn't legal (motto of the legal seafood restaurant).
What does this mean and what do many people think it means?
Many people interpret this to mean, If it isn't legal, it isn't fresh,
But actually all it really says is, If it is legal, it is fresh.
That is, Our fish are fresh. We have fresh fish.

Practice problems: Build a truth table for each proposition (1)-(12)

(1) $(P \wedge \neg Q)$
(2) $(P \vee Q) \wedge R$
(3) $\neg P \wedge Q \wedge R$ (What is the convention on order of operations here?)
(4) $(S \wedge \neg Q) \vee \neg R$
(5) $(P \vee \neg P)$
(6) $(P \wedge Q) \vee (\neg P \wedge S)$
(7) $\neg P \rightarrow Q$
(8) $Q \wedge \neg Q$
(9) $Q \vee (\neg R \rightarrow \neg P)$
(10) $(P \wedge \neg Q) \rightarrow (Q \vee \neg P)$.
(11) $\neg P \leftrightarrow Q$
(12) $Q \leftrightarrow (P \vee \neg R)$

Negate these propositions, justify each step and simplify.

(13) $(P \wedge \neg Q)$
(14) $(P \vee Q) \wedge R$
(15) $\neg P \wedge Q \wedge R$ (What is the convention on order of operations here?)
(16) $(P \wedge \neg Q) \vee \neg R$
(17) $(P \wedge Q) \vee (\neg R \wedge S)$
(18) $\neg P \rightarrow Q$
(19) $(\neg P \rightarrow Q) \rightarrow P$
(20) $(P \vee (\neg R \rightarrow Q))$
(21) $\neg P \rightarrow (\neg Q \rightarrow P)$
(22) $(P \vee \neg P)$
(23) $(P \wedge \neg P)$
(24) $(P \wedge Q) \rightarrow (R \vee \neg P)$
(25) $\neg P \rightarrow ((R \wedge Q) \rightarrow \neg S)$
(26) $(P \wedge Q) \vee (\neg R \rightarrow S)$
(27) $P \leftrightarrow Q$
(28) $\neg R \leftrightarrow S$

(29) $P \leftrightarrow (\neg Q \wedge R)$

For each pair of statements, determine if the pair of statements are logically equivalent or not. Carefully show, explain or justify your answers.

(30)	$\neg P \to Q$	$P \wedge Q$
(31)	$P \leftrightarrow Q$	$(P \to Q) \wedge (Q \to P)$
(32)	$\neg R \to (Q \to P)$	$(Q \wedge \neg P) \to \neg R$
(33)	$\neg Q \to (P \to R)$	$(Q \wedge \neg P) \to (Q \vee \neg R)$
(34)	$P \to (Q \vee R)$	$(P \wedge \neg Q) \to R$
(35)	$P \to (Q \vee R)$	$(P \wedge \neg R) \to Q$
(36)	$\neg P \vee (Q \to R)$	$(P \wedge \neg R) \to Q$

When writing these propositions be sure to remove double negatives and simplify your statements.

37. For the proposition: If the moon is not bright then the outline is clear.
    (a) Write the inverse in English.
    (b) Write the contrapositive in English.
    (c) Write the converse in English.

38. For the proposition: If the highway is narrow then trucks are not allowed.
    (d) Write the inverse in English.
    (e) Write the contrapositive in English.
    (f) Write the converse in English.

39. For the proposition: The wall is not blue and the ceiling is green.
    (g) Write it in symbolic logic.
    (h) Write its negation symbolically and simplify it.
    (i) Write its negation in English.

40. For the proposition: Either the river is deep or the trees are not high.
    (j) Write it in symbolic logic.
    (k) Write its negation symbolically and simplify it.
    (l) Write its negation in English.

41. For the proposition: If the apes make music then the lions are not quiet.
    (m) Write it in symbolic logic.
    (n) Write its negation symbolically and simplify it.
    (o) Write its negation in English.

42. For the proposition: If the lecture is not long then they wait.
    (p) Write it in symbolic logic.
    (q) Write its negation symbolically and simplify it.
    (r) Write its negation in English.

43. Consider the statement: If today is Monday then today is Tuesday, When (for what truth values) is it true and when is it false?

<u>Some solutions:</u> Please do not look at the solutions until you have worked out and written down answers to the problems above. It will be best for you to work these out before looking at any answers.
Build a truth table for each proposition

(1) $(P \wedge \neg Q)$

P	Q	$\neg Q$	$P \wedge (\neg Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(2) $(P \vee Q) \wedge R$

(3) $\neg P \wedge Q \wedge R$ (What is the convention on order of operations here?)

$\neg P$	P	Q	R	$\neg P \wedge Q$	$(\neg P \wedge Q) \wedge R)$	
F	T	T	T	F	F	
F	T	T	F	F	F	
F	T	F	T	F	F	
F	T	F	F	F	F	
T	F	T	T	T	**T**	
T	F	T	F	T	F	
T	F	F	T	F	F	
T	F	F	F	F	F	

The convention is that if the same sign appears we read and compute **left to right.** The parentheses just emphasis the convention. This is also the convention that we use when add or subtract numbers.
For example, 2 + 7 + 6 = (2+7) +6 = 13. This does not matter when we **add** numbers because addition and associative. It doesn't matter where we put the parentheses: 2 + 7 + 6 = (2+7) +6 = 2 + (7+ 6) = 13. However it DOES matter when we subtract:
15 - 7 - 6 = (15 -7) - 6 = 8 – 6 = 2. But 15 – (7 - 6) = 15 – 1 = 14.
We will learn that both "AND" and "OR" are commutative and associative so it does not matter what order we do them in as long as they are all the same operation, that is all "AND"s or all "OR's.
Notice that, order does matter when we mix + and -.
10 + 6 - 2
For this computation we must use the convention of going from left to right. So here the parentheses would be useful 10 + 6 - 2 means (10 + 6) - 2 = 14.
10 + 6 -2 DOES NOT mean 10 + (6 – 2) = 10 - 4 = 6

(4) $(S \wedge \neg Q) \vee \neg R$

(5) $(P \vee \neg P)$

(6) $(P \wedge Q) \vee (\neg P \wedge S)$

$\neg P$	P	Q	S	$P \wedge Q$	$\neg P \wedge S$	$(P \wedge Q) \vee (\neg P \wedge S)$
F	T	T	T	T	F	T
F	T	T	F	T	F	T
F	T	F	T	F	F	F
F	T	F	F	F	F	F
T	F	T	T	F	T	T
T	F	T	F	F	F	F
T	F	F	T	F	T	T
T	F	F	F	F	F	F

(7) $\neg P \rightarrow Q$

(8) $Q \wedge \neg Q)$

(9) $Q \vee (\neg R \rightarrow \neg P)$

(10) $(P \wedge \neg Q) \rightarrow (Q \vee \neg P)$.

(11) $\neg P \leftrightarrow Q$

$\neg P$	P	Q	$\neg P \rightarrow Q$	$Q \rightarrow \neg P$	$\neg P \leftrightarrow Q \equiv (\neg P \rightarrow Q) \wedge (Q \rightarrow \neg P)$
F	T	T	T	F	F
F	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	F

(12) $Q \leftrightarrow (P \vee \neg R)$

Negate these propositions, justify each step and simplify.

(13) $(P \wedge \neg Q)$

$\neg(P \wedge \neg Q) \equiv \neg P \vee \neg\neg Q \equiv \neg P \vee Q$

Demorgan's law Double negation

(14) $(P \vee Q) \wedge R$

$\neg[(P \vee Q) \wedge R] \equiv \neg(P \vee Q) \vee \neg R \equiv (\neg P \wedge \neg Q) \vee \neg R$

Demorgan's law Demorgan's law

(15) $\neg P \wedge Q \wedge R$ (What is the convention on order of operations here?)

$\neg(\neg P \wedge Q \wedge R) \equiv \neg((\neg P \wedge Q) \wedge R) \equiv \neg(\neg P \wedge Q) \vee \neg R)$

Associativity of $\wedge$ Demorgan's law

$\equiv [\neg(\neg P) \vee \neg Q)] \vee \neg R \equiv P \vee \neg Q \vee \neg R)$

Demorgan's law Double negation and associativity

(16) $(P \wedge \neg Q) \vee \neg R$

$\neg[(P \wedge \neg Q) \vee \neg R] \equiv \neg(P \wedge \neg Q) \wedge \neg\neg R \equiv (\neg P \vee \neg\neg Q) \wedge \neg\neg R$

Demorgan's law Demorgan's law

$\equiv (\neg P \vee Q) \wedge R$

Double negation (twice)

(17) $(P \wedge Q) \vee (\neg R \wedge S)$

$\neg[(P \wedge Q) \vee (\neg R \wedge S)] \equiv \neg(P \wedge Q) \wedge \neg(\neg R \wedge S)$

Demorgan's law

$\equiv (\neg P \vee \neg Q) \wedge (\neg\neg R \vee \neg S) \equiv (\neg P \vee \neg Q) \wedge (R \vee \neg S)$

Demorgan's law (twice) Double negation

(18) $\neg P \rightarrow Q$

$\neg(\neg P \rightarrow Q) \equiv \neg P \wedge \neg Q$

Negation of an implication (Nimp/and)

(19) $(\neg P \rightarrow Q) \rightarrow P$

$\neg[(\neg P \rightarrow Q) \rightarrow P] \equiv (\neg P \rightarrow Q) \wedge \neg P$ (also, $\equiv (P \vee Q) \wedge \neg P$)

Negation of an implication (Nimp/and)

(20) $(P \vee (\neg R \rightarrow Q))$

$\neg[(P \vee (\neg R \rightarrow Q))] \equiv \neg P \wedge \neg(\neg R \rightarrow Q) \equiv \neg P \wedge (\neg R \wedge \neg Q)$

Demorgan's law Negation of an implication (Nimp/and)

(21) $\neg P \rightarrow (\neg Q \rightarrow P)$ $\downarrow \leftarrow$ Nimp/and $\rightarrow \downarrow$

$\neg[\neg P \rightarrow (\neg Q \rightarrow P)] \equiv \neg P \wedge \neg(\neg Q \rightarrow P) \equiv \neg P \wedge (\neg Q \wedge \neg P)$

(22) $P \vee \neg P$ Note that this statement is always true.

$\neg(P \vee \neg P) \equiv \neg P \wedge P$ So this is always false.

(23) $P \wedge \neg P$ Note that this statement is always false.

$\neg(P \wedge \neg P) \equiv \neg P \vee P$ So this is always true.

(24) $(P \wedge Q) \rightarrow (R \vee \neg P)$

$\neg[(P \wedge Q) \rightarrow (R \vee \neg P)] \equiv (P \wedge Q) \wedge \neg(R \vee \neg P) \equiv (P \wedge Q) \wedge (\neg R \wedge \neg\neg P)$

Nimp/and Demorgan's law

Use Double negation to obtain $\equiv (P \wedge Q) \wedge (\neg R \wedge P)$

(25) $\neg P \to ((R \wedge Q) \to \neg S)$

$\neg[\neg P \to ((R \wedge Q) \to \neg S)] \equiv \neg P \wedge \neg((R \wedge Q) \to \neg S)$

Nimp/and

$\equiv \neg P \wedge ((R \wedge Q) \wedge \neg\neg S) \equiv \neg P \wedge ((R \wedge Q) \wedge S)$

Nimp/and Double negation

(26) $(P \wedge Q) \vee (\neg R \to S)$

$\neg[(P \wedge Q) \vee (\neg R \to S)] \equiv \neg(P \wedge Q) \wedge \neg(\neg R \to S)$

Demorgan's law

$\equiv (\neg P \vee \neg Q) \wedge (\neg R \wedge \neg S)$

Demorgan's law and Nimp/and

(27) $P \leftrightarrow Q$

$\neg(P \leftrightarrow Q) \equiv \neg((P \to Q) \wedge (Q \to P)) \equiv \neg(P \to Q) \vee \neg(Q \to P))$

Definition of biconditional (iff) Demorgan's law

$\equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$

Nimp/and (twice)

(28) $\neg R \leftrightarrow S$

$\neg(\neg R \leftrightarrow S) \equiv \neg((\neg R \to S) \wedge (S \to \neg R)) \equiv \neg(\neg R \to S) \vee \neg(S \to \neg R))$

Definition of biconditional (iff) Demorgan's law

$\equiv (\neg R \wedge \neg S) \vee (S \wedge \neg\neg P)) \equiv (\neg R \wedge \neg S) \vee (S \wedge P))$

Nimp/and (twice) Double negation

(29) $P \leftrightarrow (\neg Q \wedge R)$

$\neg(P \leftrightarrow (\neg Q \wedge R)) \equiv \neg((P \to (\neg Q \wedge R)) \wedge ((\neg Q \wedge R) \to P))$

Definition of biconditional (iff)

$\equiv \neg(P \to (\neg Q \wedge R)) \vee \neg((\neg Q \wedge R) \to P)$

Demorgan's law

$\equiv P \wedge \neg(\neg Q \wedge R) \vee (\neg Q \wedge R) \wedge \neg P \equiv P \wedge (Q \vee \neg R) \vee (\neg Q \wedge R \wedge \neg P)$

Nimp/and (twice) Demorgan's law and Double negation

For each pair of statements, determine if the pair of statements are logically equivalent or not. Carefully show, explain or justify your answers.

(30) $\neg P \to Q$ $\quad P \wedge Q$ Not equivalent Try P-T Q-F

(31) $P \leftrightarrow Q$ $\quad P \to Q) \wedge (Q \to P)$ Equivalent Def'n of $\leftrightarrow$

(32) $\neg R \to (Q \to P)$ $\quad (Q \wedge \neg P) \to \neg R$ Not equivalent R-T,Q-T,P-F

(33) $\neg Q \to (P \to R)$ $\quad (Q \wedge \neg P) \to (Q \vee \neg R)$ Not equivalent

(34) $P \to (Q \vee R)$ $\quad (P \wedge \neg Q) \to R$ Equivalent

(35) $P \to (Q \vee R)$ $\quad (P \wedge \neg R) \to Q$ Equivalent

(36) $\neg P \vee (Q \rightarrow R)$ $(P \wedge \neg R) \rightarrow Q$ Not equivalent Q-F,P-T,R-F7

37. For the proposition: If the moon is not bright then the outline is clear.

(a) Write the inverse in English.

If the moon is bright then the outline is not clear.

(b) Write the contrapositive in English.

If the outline is not clear then the moon is bright.

(c) Write the converse in English.

If the outline is clear then the moon is not bright.

38. For the proposition: If the highway is narrow then trucks are not allowed.

(d) Write the inverse in English.

If the highway is not narrow then trucks are allowed.

(e) Write the contrapositive in English.

If trucks are allowed then the highway is not narrow.

(f) Write the converse in English.

If trucks are not allowed then the highway is narrow.

39. For the proposition: The wall is not blue and the ceiling is green.

Legend: Let B represent "The wall is blue" & G represent "The ceiling is green."

(g) Write it in symbolic logic. $\neg B \wedge G$

(h) Write its negation symbolically and simplify it.

$\neg(\neg B \wedge G) \equiv \neg\neg B \vee \neg G \equiv B \vee \neg G$

Demorgan's Law Double negation

(i) Write its negation in English. Either the wall is blue or the ceiling is not green.

40. For the proposition: Either the river is deep or the trees are not high.

Legend:

Let D represent "The river is deep" and H represent "The trees are high."

(j) Write it in symbolic logic. $D \vee \neg H$

(k) Write its negation symbolically and simplify it.

$\neg(D \vee \neg H) \equiv \neg D \wedge \neg\neg H \equiv \neg D \wedge H$

Demorgan's Law Double negation

(l) Write its negation in English.

The river is not deep and the trees are high.

41. For the proposition: If the apes make music then the lions are not quiet.

Legend:

Let M represent "The apes make music" and Q represent "The lions are quiet."

(m) Write it in symbolic logic. $M \rightarrow \neg Q$

(n) Write its negation symbolically and simplify it.

$\neg(M \rightarrow \neg Q) \equiv M \wedge \neg\neg Q \equiv M \wedge Q$

Negation of an implication Double negation

(o) Write its negation in English.

The apes make music and the lions are quiet.

42. For the proposition: If the lecture is not long then they wait.
    (p) Write it in symbolic logic.
    (q) Write its negation symbolically and simplify it.
    (r) Write its negation in English.

43. Consider the statement: If today is Monday then today is Tuesday, When (for what truth values) is it true and when is it false? Consider the cases for the truth value of “today is Monday”. Do they influence the truth value of “today is Tuesday”? Remember how implication works!

VALID ARGUMENTS

Definition 1.12: An **argument** is a finite sequence of propositions, called premises, followed by a proposition, called the conclusion.

Symbolic examples of some arguments:

$$\begin{array}{l} P\to Q \\ P \\ \hline Q \end{array} \qquad \begin{array}{l} P\to Q \\ Q \\ \hline P \end{array} \qquad \begin{array}{l} P\to Q \\ \neg Q \\ \hline \neg P \end{array} \qquad \begin{array}{l} \neg P\to Q \\ R\to \neg Q \\ R \\ \hline P \end{array} \qquad \begin{array}{l} Q\to P \\ \neg R\to \neg P \\ \hline Q\to R \end{array} \qquad \begin{array}{l} P\to Q \\ \neg P \\ \hline \neg Q \end{array} \qquad \begin{array}{l} P\wedge Q \\ P\to R \\ \hline R \end{array}$$

$$\begin{array}{l} P\to Q \\ Q\to R \\ \hline P\to R \end{array} \qquad \begin{array}{l} P\vee \neg S \\ \neg Q\wedge R \\ S\to Q \\ \hline P \end{array} \qquad \begin{array}{l} (R\vee S)\to P \\ \neg Q\wedge R \\ \hline \neg P \end{array} \qquad \begin{array}{l} \neg S\vee P \\ \neg P\vee Q \\ \neg Q \\ \hline \neg S \end{array} \qquad \begin{array}{l} P\to \neg Q \\ Q\to (R\vee S) \\ \hline P\to (R\vee S) \end{array} \qquad \begin{array}{l} P\vee R \\ P\to R \\ \hline R \end{array}$$

Definition 1.13: An argument is **valid** unless there is a truth assignment in which all the premises are true and the conclusion is false. If an argument is not valid, then we say it is **invalid**.

One way to show that an argument is valid is to build the entire truth table. First we "AND" all of the premises together, put brackets around all of that and ask if it "IMPLIES" the conclusion.
If this long proposition is **always** true, then the argument is valid.
That is, if we get all T's in the last column of our truth table (under this statement), then the argument is **valid**.
This gives us another way to determine if an argument is valid or not
If we get at least one F, then the argument is **invalid**.
Example: $((P\to Q) \wedge P) \to Q$).

P	Q	$P\to Q$	$(P\to Q)\wedge P$	$(P\to Q)\wedge P)\to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since there is a T in EACH row of the final column of the truth table, under $((P\to Q) \wedge P) \to Q$), then by definition this argument is valid.

Example: $((P \to Q) \land \neg P) \to \neg Q$).

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$(P \to Q) \land \neg P$	$(P \to Q) \land \neg P) \to \neg Q$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	**F**
F	F	T	T	T	T	T

Since there is an F in at least one row of the final column of the truth table, under $(P \to Q) \land \neg P) \to \neg Q)$, then by definition this argument is NOT valid (or invalid).
Can you describe why this argument seems invalid? Can you give an example of an argument of this form which illustrate that the reasoning is incorrect?

There are at least 3 ways to determine whether an argument is valid or invalid.
First: As we did a moment ago, we could <u>build the entire truth table</u> for the statement with all of the premises “and”ed together and “implying” the conclusion. We really do NOT want to do this. It is too long and there are better ways.
Second: We could make the conclusion false and attempt to make all the premises true. If we can do this the argument must be invalid. WHY?
On the other hand, if it is impossible to make the conclusion false while all the premises are true then the argument is valid. We might call this the method of the <u>mini truth table</u>. Using this second method, it is usually easier to show that an argument is invalid and a little more difficult to show that the argument is valid.

Example: Let’s show that this argument

$$\begin{array}{l} \neg P \lor R \\ \underline{P \to R} \\ \quad R \end{array}$$

is not valid by using a small truth table.

$$\begin{array}{l} \neg P \lor R \\ \underline{P \to R} \\ R \end{array}$$

Make a small truth table and make the conclusion false:

P	R
	F

Now, the question is, Can we make all the premises true, with the assumption that the conclusion is false? That is, making R false, can we now assign truth values to the other proposition(s) so that each premise is true?

In this case the answer is YES.
Since R is false, the only way we can make the first premise, $\neg P \lor R$, true is to make $\neg P$ true. So P should be F. Now the first premise is T. WHY?

So we fill in a F for P in our mini truth table:

P	R
F	F

Now the second premise, P$\rightarrow$R, is true when P is F. WHY?
Thus we have found a truth assignment that makes the conclusion false and all the premises true, so by definition the argument is invalid.

$\neg P \vee R$	$\neg F \vee F$ is	**T**
$P \rightarrow R$	$F \rightarrow F$ is	**T**
R		**F** Thus the argument is INVALID.

Let's try the first method on this same argument and see what happens.

$\neg P$	P	R	$\neg P \vee R$	$P \rightarrow R$	$(\neg P \vee R) \wedge (P \rightarrow R)$	$((\neg P \vee R) \wedge (P \rightarrow R)) \rightarrow R$
F	T	T	T	T	T	T
F	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	T	**F**

Note that the **F** in the final column shows that the argument is invalid.
And, in fact, that **F** in the final column came in the row with P False and R False, just as in our mini truth table. Our thinking, that built the mini truth table, helped us to find the exact place where the argument would fail to be valid. That is easier then mechanically building the entire truth table and checking every possibility.
In your own words describe the _only_ way we could get an **F** in the final column of a truth table for an argument.

We might think that we could apply this method to *any* argument to show it is invalid. But, if the argument is valid we will not be able to make all the premises true while the conclusion is false. Let's try this on the argument

$$\begin{array}{l} P \rightarrow Q \\ \underline{P\quad} \\ Q \end{array}$$

We can make the conclusion false by letting Q be False.

$$\begin{array}{ll} P \rightarrow Q & \\ \underline{\;P\quad} & \\ Q & \textbf{F} \end{array}$$

P	Q
	F

And the only way the second premise could be True is if we make P true.

$$\begin{array}{ll} P \rightarrow Q & \\ \underline{\;P\quad} & \textbf{T} \\ Q & \textbf{F} \end{array}$$

P	Q
T	F

But now, when we look at the first premise, $P \to Q$, we see that it MUST BE false because True$\to$False is FALSE.
Because there were no other choices, we found that it is impossible to make all the premises true while the conclusion is false. Therefore this argument is valid.

Practice: Now apply the mini truth table method to each of the other arguments listed above.

The third method for determining whether an argument is valid is to use known equivalences and known valid arguments to construct the conclusion from the premises. This will tell us that the argument is valid. This is often a useful way to see the structure of the argument and what statements are being used to prove other statements.

Remember: $P \wedge Q$ is true iff both P and Q are true
$P \vee Q$ is false iff both P and Q are false
$P \to Q$ is false iff P is true and Q is false
Note: $P \to Q$ can be described as P is a sufficient condition for Q
or as Q is a necessary condition for P.

Some equivalences: $\neg\neg P \equiv P$ Double Negation
$P \to Q \equiv \neg Q \to \neg P$ Contrapositive
$P \to Q \equiv \neg P \vee Q$ Imp-Or (an implication becomes an Or statement)
$\neg(P \to Q) \equiv P \wedge \neg Q$ Negation of an implication
$(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P)$ Biconditional or iff (if and only if)
$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$
$P \to (Q \vee R) \equiv (P \wedge \neg R) \to Q$ Disjoined Conclusions

DeMorgan's Laws: $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

$P \vee Q \equiv Q \vee P$ $\vee$ (or) is a commutative operation
$P \wedge Q \equiv Q \wedge P$ $\wedge$ (and) is a commutative operation
NOTE: $\to$ (implication) is not commutative
$P \to Q$ is NOT equivalent to $Q \to P$ (Can you prove this? Do it now.)

$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $\vee$ (or) is an associative operation
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ $\wedge$ (and) is an associative operation
NOTE: $\to$ (implication) is not associative
$(P \to Q) \to R$ is NOT equivalent to $P \to (Q \to R)$
(Can you prove this? Do it now.)

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ "and" distributes over "or"
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ "or" distributes over "and"

Compare this to multiplication and addition.
Does multiplication distribute over addition?
Is $5 \cdot (4+3) = (5\cdot 4) + (5\cdot 3)$?
Does addition distribute over multiplication?
Is $5 + (4\cdot 3) = (5+4)\cdot(5+3)$?

Here is the truth table for the logical equivalence **Disjoined Conclusions**

$$P\to(Q\vee R) \equiv (P \wedge \neg Q)\to R$$

$\neg Q$	P	Q	R	$Q\vee R$	$(P \wedge \neg Q)$	$P\to(Q\vee R)$	$\equiv (P \wedge \neg Q) \to R$
F	T	T	T	T	F	T	T
F	T	T	F	T	F	T	T
T	T	F	T	T	T	T	T
T	T	F	F	F	T	F	F
F	F	T	T	T	F	T	T
F	F	T	F	T	F	T	T
T	F	F	T	T	F	T	T
T	F	F	F	F	F	T	T

We will want to use Disjoined Conclusions later. It is a powerful proof technique. For now, just recognize that these two propositions are equivalent: $P\to(Q\vee R) \equiv (P \wedge \neg Q)\to R$

<u>**Five valid arguments** and their names</u>

Modus Ponens
$P\to Q$
P
$\overline{Q}$

Modus Tollens
$P\to Q$
$\neg Q$
$\overline{\neg P}$

Disjunctive Syllogism (DS)
$P\vee Q$, $\neg P$, therefore Q or $P\vee Q$, $\neg Q$, therefore P

$$\begin{array}{l} P\vee Q \\ \neg P \\ \hline Q \end{array} \quad \text{or} \quad \begin{array}{l} P\vee Q \\ \neg Q \\ \hline P \end{array}$$

Transitivity
$$\begin{array}{l} P\to Q \\ Q\to R \\ \hline P\to R \end{array}$$

Dilemma
$$\begin{array}{l} P\vee Q \\ P\to R \\ Q\to S \\ \hline R\vee S \end{array}$$

Here are some examples of arguments. Look at each argument, build a legend, use letters to stand for the statements, write out symbolic representations of these arguments and determine if the argument is valid or not. Be sure to justify each step with a valid equivalence or argument. Good Practice!

(1) If we go north then we ski.
We go north.

We ski.

(2) If I do not sleep then I do not study.
I do not sleep.

I do not study.

(3) If we are late we will miss lunch.
We miss lunch.

We are late.

(4) If we are late we will miss lunch.
We didn’t miss lunch.

We are not late.

(5) Either you pay your bill or you don’t graduate.
You graduate.

You paid your bill.

(6) If she studies a lot she will do well.
She will not be happy if she does not do well.

If she studies a lot she will be happy.

(7) If the temperature reaches –22° F, gypsy moth eggs will freeze.
If we have gypsy moths in the spring, their eggs did not freeze.
The temperature reached –22° F.

There will be no gypsy moths this spring.

(8) If I do my job I will be rewarded.
Either I do my job or I go hiking.
If I go hiking then I will feel good.

Either I will feel good or I will be rewarded.

(9) If we pack a lunch then we must buy bread.
If we don’t pack a lunch then we don’t go on a picnic.
If we buy bread then we must go to the store.

We must go to the store.

(10) Hot, humid weather favors thunderstorms.
Hot, humid weather favors bacterial growth.
Bacterial growth causes milk to sour.
Thunderstorms cause milk to sour.

(11) Either I don't go to college or I get a good job.
Either I make a lot of money or I don't get a good job.
If I do not work hard then I do not make a lot of money.
I go to college.
I do not work hard.

(12) If my car breaks down then I will not go to New York.
If I get $500 then I will go to New York.
My car breaks down.
I didn't get $500.

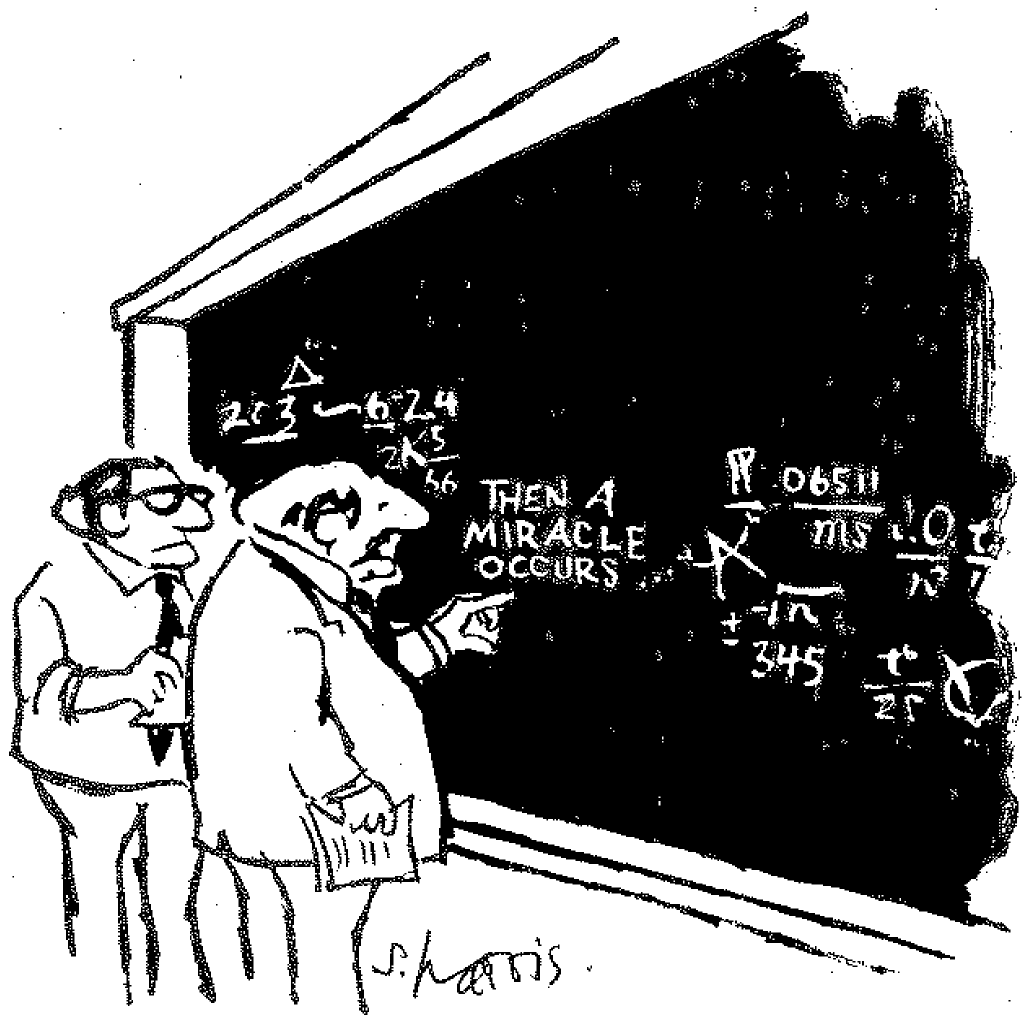
(13) All koalas are shy.
All shy animals are friendly.
All koalas are friendly.

(14) Either my uncle is an art critic or muffins are unhealthy.
If aliens write poetry, then muffins are unhealthy and my uncle is an art critic.
But it is assuredly not true both that muffins are unhealthy and aliens do not write poetry.
Hence, muffins are unhealthy and my uncle is an art critic.

(15) If the plants are healthy then the trees are tall.
The soil is not rich or the days are short.
If the days are not short then the sun shines.
The trees are tall and the soil is rich.

(16) Either the sun is shining or the road is not rough.
If the quality is good then product is not plastic.
Either the product is plastic or the sun is not shining.
The road is rough.
The quality is not good.

(17) If I am in New England then it is not the case that both it is winter and there is no snow.
Either it is not the case that both it is winter and there is no snow or if I am not inside then I am cold.
I am inside and I am in New England.
I am not cold.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Definition 1.14: Any statement that is always true no matter what truth values the individual propositions have is called a <u>tautology</u>.

Every *valid* argument, when written as a single statement, is a tautology. For example, $((P \to Q \wedge \neg Q) \to \neg P)$ is always true no matter whether P is true or false and no matter whether Q is true or false. So the statement is a tautology. And if we have $P \to Q$ and $\neg Q$ then we may always conclude $\neg P$. Thus we can always use tautologies because they are always true.

Other <u>tautologies</u>:

(a) Any logical equivalence, $\equiv$, is a tautology.
For example, since $\neg\neg P \equiv P$ is always true,
So any time we have $\neg\neg P$ we may replace it with P.

(b) Strange but useful tautologies:

$P \rightarrow (P \vee Q)$ $\quad$ $Q \rightarrow (P \vee Q)$

$(P \wedge Q) \rightarrow P$ $\quad$ $(P \wedge Q) \rightarrow Q$

$(P \vee P) \rightarrow P$

Note that these can be written as arguments.
For example: $P \rightarrow (P \vee Q)$ is the valid argument.
This just says, If we have P, then we either P or Q.
We can write this in the form of an argument:

$$\frac{P}{P \vee Q}$$

Also $(P \wedge Q) \rightarrow P$ is a valid argument.
This just says, If we have P and Q then we have P.
Written as an argument:

$$\frac{P \wedge Q}{P}$$

Exercise: Prove that each of the five strange but useful tautologies above, when written as an argument, is a valid argument.
Describe in simple English terms why each of these is valid.

Observe that the following are <u>NOT</u> tautologies so we can NOT use them in a proof.

$(P \vee Q) \rightarrow P$ $\quad$ $(P \vee Q) \rightarrow Q$

$P \rightarrow (P \wedge Q)$ $\quad$ $Q \rightarrow (P \wedge Q)$

Exercise: Prove that each of the four statements above, when written as an argument, is not a valid argument
Describe in simple English terms why each of these is not valid.

<u>**Fallacies**</u>: Fallacies are well known INVALID arguments. People often mistake them for valid arguments because they "seem like" they are true. However we can test them and see that they are not valid

Fallacy of affirming the consequent

$$\frac{\begin{array}{l} P \rightarrow Q \\ Q \end{array}}{P}$$

Fallacy of denying the antecedent

$$\frac{\begin{array}{l} P \rightarrow Q \\ \neg P \end{array}}{\neg Q}$$

<u>Prove</u> that the above two arguments are not valid.

Techniques for determining when an argument is valid and when it is invalid.

It is helpful to make a legend, that is, we can use letters to represent statements. For example: Let L represent "we are late"
M represent "we will miss lunch"

So the argument:

(3) If we are late we will miss lunch.
We miss lunch.

We are late.

becomes

$$\frac{\begin{array}{c} L \rightarrow M \\ M \end{array}}{L}$$

This argument has a form similar to MP (Modus Ponens) but not quite. It is also similar to MT (Modus Tolens) but not quite. It is not the exact same form as either one. Let's show it is invalid.

First, we make the conclusion false (remember, to show an argument is invalid we must make the conclusion false while making all the premises true). We make the conclusion false by letting L be False (F).

A little chart might help:

L	M
F	

Now try to make each premise True. The only way M (as a premise) could be True, is if it is True. So put a T in the chart.

L	M
F	T

And now look at the other premise.
When L is False, then $L \rightarrow M$ is True by material implication (our definition of implication said that- False implies anything is automatically True). Since we CAN make all the premises True while the conclusion is False. $L \rightarrow M$ is True
M is True
Yet L is False.

Thus this argument is not valid.

Note that this is the fallacy of affirming the consequent.

Some solutions: Look at each argument, use letters to represent the statements, then determine if the argument is valid or not.
Let N represent "we go north" and S represent "we ski". The argument:

(1) If we go north then we ski.		$N \to S$
We go north.	becomes	N
-----------------------------------		--------------
We ski.		S

which is exactly the form of MP, so is a valid argument.

Similarly,

(2) If I do not sleep then I do not study.		$\neg S \to \neg D$
I do not sleep.	becomes	$\neg S$
-------------------------------------		---------------
I do not study.		$\neg D$

which is again exactly the form of MP, since $\neg S$ is one premise and $\neg S \to \neg D$ is the other premise, so by MP we conclude $\neg D$. Thus, this is a valid argument.

---

1) Valid by MP 2)Valid by MP 3)Not valid. Let L be F and M be T.
4) Valid by MT 5) Valid by DS 6)Not valid. Let S-T, W-T, H-F

6) Consider $[(S \to W) \wedge (\neg W \to \neg H)] \to (S \to H)$. To make this implication false, we try to make the hypotheses true while making the conclusion false. The only way that the conclusion, $S \to H$, could be false would be if S were true and H were false. Now to make the premise, $S \to W$, true, since S must be true (to keep the conclusion false) we make W true. Note that $\neg W \to \neg H$ is equivalent to $H \to W$, by contrapositive. So now we show that $H \to W$ is true by making H false.

	S	H	W	
This truth assignment				
makes the conclusion false	T	F	T	while keeping the

premises true, thus demonstrating that the argument is not valid.

7) Note that it is probably not a good idea to represent statements by T of F. So, "the temperature reaches -22° F" should be represented by TE or TP or TEMP but not by T, because that would create a confusion with T for true.

$TEMP \to EF$
$G \to \neg EF$
$TEMP$

$\neg G$

Now, TEMP together with TEMP$\rightarrow$ EF gives EF by Modus Ponens (MP)
Also, G$\rightarrow \neg$ EF $\equiv$ EF $\rightarrow \neg$ G by contrapositive
Finally, EF (obtained above) and EF $\rightarrow \neg$ G together give $\neg$ G by MP.
Thus the argument is valid.
8) Valid by dilemma 9) Not valid. We attempt to show this argument is invalid. First, to make the conclusion false we let S be F. Then, to make the premise, B $\rightarrow$ S, true, we must label B as F. Working, in this way, since we have L$\rightarrow$ B , then L must be F to make the whole statement true. Finally, since L must be false then P must also be false to make $\neg$ L$\rightarrow \neg$ P true. Now we have found a truth assignment which makes the conclusion false and all the premises true, so the argument is not valid.

10) H$\rightarrow$ S
H$\rightarrow$ B
B$\rightarrow$ M

S$\rightarrow$ M

We can obtain H$\rightarrow$ M by transitivity, but not S$\rightarrow$M. So try to show it is not valid by making S$\rightarrow$ M false and all the hypotheses true. How could S$\rightarrow$ M be false? S would have to be true and M would have to be false. Working on trying to make the premise, B$\rightarrow$M, true, since M is false, we must make B false. B false forces H to be false. This truth assignment makes all the premises true and the conclusion false, so the argument is not valid.
11) Not valid. Use C, GJ, M, WH all true.

12) C$\rightarrow$ $\neg$ NY
FIVE$\rightarrow$ NY $\equiv$ $\neg$ NY $\rightarrow \neg$ FIVE by contrapositive
C

$\neg$ FIVE

So, C and C$\rightarrow \neg$ NY give us $\neg$ NY. $\neg$ NY and $\neg$ NY $\rightarrow \neg$ FIVE give us $\neg$ FIVE. Thus the argument is valid.
Note that there are several ways to show that this argument is valid:
Contrapositive plus 2 applications of MP
MP plus MT
Contrapositive plus transitivity plus MP

13) If x is a koala then x is shy.
If x is shy then x is friendly
If x is a koala then x is friendly.   Valid by transitivity.

---

14) Either my uncle is an art critic or muffins are unhealthy.
If aliens write poetry, then muffins are unhealthy and my uncle is an art critic.
But it is assuredly not true both that muffins are unhealthy and aliens do not write poetry.

Hence, muffins are unhealthy and my uncle is an art critic.

$U \vee \neg M$
$A \rightarrow (\neg M \wedge U)$
$\neg (\neg M \wedge \neg A)$

$\neg M \wedge U$

There are three ways that the conclusion could be false:
(i) both $\neg M$ and U could be false (so M is True and U is False), or
(ii) $\neg M$ could be False and U could be True (so M is True and U is True), or
(iii) $\neg M$ could be True and U could be False (so M is False and U is False).
Now let's be careful here. What would we need to do to show that the argument is valid? We would have to make sure that it was IMPOSSIBLE to make all the premises true while the conclusion was false. To show that the argument is invalid we would only need to finds one truth assignment that made all the premises true while the conclusion was false. Suppose we tried case (iii) where M False and U False (remember this makes the conclusion false). Since M is False, then $\neg M$ is True, so the first premise is true. Since M is False, then $\neg M$ is True and U is False so the conclusion of the second premise is False. Thus the only way the second premise could be true is if A is False. But then the third premise has $\neg M$ True and $\neg A$ True, so $(\neg M \wedge \neg A)$ is True and thus, $\neg (\neg M \wedge \neg A)$ must be False. Therefore we cannot make all the premises true when the conclusion is False UNDER THIS ASSIGNMENT. Does this make the argument valid? NO. We must show that it is impossible to make the premises True while the conclusion is False under ANY truth value assignment. So we must test the other possibilities. Suppose we try case (i) where M is True and U is False.Here the first premise cannot be True.
So let's try case (ii) where M is True and U is True. Since U is True then the first premise is True. Since M is True then $\neg M$ is False, so the conclusion of the second premise is False Thus the only way the second premise could be True is if A is False. So making A False makes the second premise True. Now in the third premise, since $\neg M$ is False, $(\neg M \wedge \neg A)$ is False and

thus, $\neg(\neg M \wedge \neg A)$ must be True. Therefore we have made all three premises True while the conclusion ids False, so the argument is INVALID.

(15) Note that the conclusion is an "and" statement.
The trees are tall **and** the land is rich.
When is an "and" statement False?
What does this mean in terms of showing whether the argument is Valid or Invalid?

(17) $N \to \neg(W \wedge \neg S)$
$\neg(W \wedge \neg S) \vee (\neg I \to C)$
$I \wedge N$
______
$\neg C$ Invalid

Quantifiers

Many times when we use a natural language like English there is information implied in a sentence, or details that are assumed from the situation or context. One example which has been carefully studied is the following scenario: Some people go into a restaurant, order lunch, sit and talk for 45 minutes, then get up, pay the bill and leave the restaurant. Did the people eat anything? The words never say that the people ate anything. Yet most people, when told this story, would say that the people did actually eat, even though the story never explicitly says that.

Similar situations occur frequently in mathematics; so, as always, we need to be very careful. We want to understand what ideas or pieces of information are contained in our statements. It is very important that we write down exactly what we mean by a statement without any hidden assumptions.

Consider a sentence like: $x > 6$.
This is not a statement since it is neither true nor false. We call it an <u>open sentence</u> because its truth value depends on a variable, so the truth value is open.

If we replace the variable with some number like $x=8$ or $x=51$ the open sentence would become a true statement: $8 > 6$, $51 > 6$. These are true.

If we replace the variable with some number like $x= 4$ or $x= -11$, the open sentence would become a false statement: $4 > 6$, $-11 > 6$. These are false.

But suppose we wanted to say: there is some number, x, such that x > 6. Or, there exists at least one number, x, such that x > 6.
These statements assert the existence of at least one number which is greater than 6. Since this is clearly true, this is a true statement.
There is a nice notation for this idea.

Notation: The Existential Quantifier, denoted by $\exists$, can be read as "**there exists**", "**for some**", or "**there exists at least one**".
"There is an even prime number."
can be written as
$\exists$n such that n is even and n is a prime number.

"Some even numbers are prime."
can be written as
$\exists$n such that n is even and n is a prime number.

"There is at least one natural number that is greater than 15."
can be written as
$\exists$m such that m is a natural number and m is greater than 15.

Because of the way the English language is structured we often add the symbol, $\ni$, and read it as "such that". This just makes it easier to read the sentence in a nice way, but it has no logical content or importance.
For example, $\exists$n $\ni$ n is an even prime number can be read as:
$\Uparrow$ $\Uparrow$ $\Uparrow$
There exists an n such that n is an even prime number.

For example, $\exists$x $\ni$ (x>15) can be read as:
$\Uparrow$ $\Uparrow$ $\Uparrow$
There exists an x such that x is greater than 15.

We could also read this statement as: "There exists at least one x such that x is greater than 15", or "For some x, x is greater than 15".
This notation, $\ni$, which we read as "such that", is not essential or even necessary. It contains no information about the proposition. It merely allows us to read the statement a little more easily in English. Thus if you left it out and wrote: $\exists$x (x > 15), that would be fine. We could just say it means: "There is a number greater than 15."

Statements that begin with an existential quantifier, $\exists$, are saying that there is a number (or object) of the type described.
Consider this statement: $\exists n \ni \left(n^2 = 1\right)$.

It says that there exists a number n such that n squared equals 1. Note that there are two numbers whose square is 1 (n could be 1 or -1).
It is important to recognize that all the existential quantifier says is that there is <u>at least</u> one number of the type described. There may be more or there may be only one. Observe that there is only one number that satisfies the statement $\exists n \ni n$ is an even prime number. (What is it?)
Yet, there are infinitely many numbers that satisfy the statement, $\exists x\ (x > 15)$.

If we want to specifically indicate *where* we are looking for the numbers or objects that we are writing about, we may use the notation, $\in$, which we may read as " is an element of" or "is a member of" or "in the set of". This tells us what set of numbers or objects we can look into to find the number or object that "exists". Please be careful: note the difference between the symbol $\in$ and the symbol $\ni$. $\in$ is an important symbol which tells us that a number or an object is a member of a set.

Example:

$\exists n \in$ natural numbers $\ni$ n is an even prime number

can be read as:

There exists a number n which is an element (or member) of the set of natural numbers such that n is an even prime number, or
There exists a number n in the set of natural numbers such that n is an even prime number. (The natural numbers are 1,2,3,4,5,… .)
This is important because some set may contain a number that makes the statement true and yet some other set may not contain a number that makes the statement true.
Consider this statement: $\exists m \ni (m+5=0)$.
Now this statement is false IF we only look at the natural numbers since there is no natural number which, when added to 5 will give us 0.
So, $\exists m \in$ natural numbers $\ni (m+5=0)$ is false.
But, $\exists m \in$ real numbers $\ni (m+5=0)$ is true since there is a real number which when added to 5 equals 0, namely -5.

Therefore it is often important to indicate the set (sometimes called the domain of discourse or universe) from which we can select the possible numbers to make the statement true.

Another example: The proposition, $\exists x \ni (x^2 = 7)$, depends on which set of numbers we are talking about. If we talking about the real numbers, it is true; however, if we are talking about the natural numbers or the rational numbers, it is not true. If we want to specifically indicate which numbers are allowed, we would use our $\in$ notation: $\exists x \in$ real numbers $\ni (x^2 = 7)$. This means, there exists an x in the set of real numbers such that $(x^2 = 7)$.
or, for some real number x, $(x^2 = 7)$.

Now we want of look at a different situation.
Many times when we see a sentence in mathematics such as $x^2 \geq 0$,
we really mean, **For any number x**, $x^2 \geq 0$, or **For every number x**, $x^2 \geq 0$.
This means we are saying, *no matter* what number you pick for x, when you square it, you get a number that is 0 or larger. This is not saying that there exists at least one. It is saying that it is true for **all** numbers.
This kind of quantifier is called a **Universal Quantifier.**

Notation: The Universal Quantifier, denoted by an upside down letter A, $\forall$, can be read as "**for every**", "**for all**", or "**for each**".

So we could write the above statement as,

$\forall x \quad (x^2 \geq 0)$ and read it as

$\Uparrow \qquad \Uparrow$

For every x   x squared is greater than or equal to 0.

The statement, If x >5 then x^2 > 25, really means
For every x, if x >5 then x^2 > 25 and is written $\forall x$ (x >5 $\rightarrow$ x^2 > 25).
Observe that we do not need a "such that" when we use a universal quantifier. This is just a quirk of English.

As before, if we want to specifically indicate the domain (what set of numbers we are talking about), then we would write

$$\forall x \in \text{real numbers } (x^2 \geq 0).$$

Note that: $\forall x \in$ real numbers $(x > 0)$ is false,
while, $\forall x \in$ natural numbers $(x > 0)$ is true.

In practice we sometimes add the quantifier at the end of the statement instead of at the beginning. We might say $x^2 \geq 0$ for all real numbers,
or
$a^2+b^2 = c^2$ for all right triangles. For our purposes we will attempt to write the quantifiers up front so that we can see what the restrictions are right away.
Thus we write: $\forall x \in$ real numbers $(x^2 \geq 0)$,
$\forall$ right triangle $(a^2+b^2 = c^2)$.

We can now say that an open sentence has been "quantified" if it has an existential or a universal quantifier in front of it.

We can even have a variety of quantifiers in front of a sentence.

In general, if we say P is some proposition then we can write

$$\forall x \forall y \ \ P.$$

We read this as: For every x and for every y the proposition P is true.

To show that $\forall x \forall y \ P$ is true, we must show that it is true for every x and for every y. To show that it is false we only need to find one x for which it isn't true or one y for which it isn't true.

To show that $\forall x \exists y \ P$ is true we must show that for each x there is some y which makes it true. It does not need to be the same y for each x. Just, given an x, we need to find some y which will work for that x to make P true. To show that it is false we only need to find one x so that no matter what y we use, it isn't true.

Let's look at a quantified proposition: $\forall x \exists y \ni (x^2 + y = 1)$
What does this proposition say?
It says that for any x there is a y such that $x^2 + y = 1$.
If this is true, this means that no matter what x you pick there must exist a y which makes $x^2 + y = 1$ true.
Is it true or false? How can we show it?

Here is how we might start to think about writing a proof.
Start by letting x be any number. Since we are assuming we are working in the real numbers unless otherwise indicated, then we can write

Let x be any real number.

Now the question is,
Can we find a y such that no matter what x we picked first (x is any real number) we will be able to show that $x^2 + y = 1$?
Well, does a y exist for any x that was chosen?
Let's solve for y to get $y = -x^2 + 1$. Will this work for any x ?
Try it! So $x^2 + y = x^2 - x^2 + 1 = 1$.
We must also check that no matter what number we put in for x we will get a real number y. But that is okay here since, given any real number for x, we can square it and then take its negative and then add 1, so $-x^2 + 1$ is always a real number when x is a real number.
Now, for practice, write out this proof in a clear and succinct argument without all the extra words above.

Consider $\forall x \exists y \ni (y = \frac{1}{x})$

Here, it is not true that for any possible x we can find a y such that $y = \frac{1}{x}$.

Why? Explain why this statement is false.

Thus, $\forall x \exists y \ni (y = \frac{1}{x})$ is false. Here is a counterexample, let $x = 0$. There is no real number y such that $y = \frac{1}{x}$ when $x = 0$.

To prove or disprove quantified statements we may use the following chart as a guide.

	Universal Statements($\forall$)	Existential Statements($\exists$)
To prove True	We need a general proof.	We need one single example.
To prove False	We need one single counterexample.	We need a general proof.

Negating Quantified Statements

What is the negation of the statement: "There exists a blue giraffe" ?
Let B(x) be the statement that that x is blue
Let G(x) be the statement that that x is a giraffe.
Then, $\exists x\ (G(x) \wedge B(x))$, says, There exists a blue giraffe .
What is its negation?
There does not exist a blue giraffe.
Or, For every object either it is not a giraffe or it is not blue.
What does this look like symbolically?

$\forall x\ (\neg G(x) \vee \neg B(x))$.

Thus, to negate an existentially quantified statement we change the quantifier to a universal quantifier and then negate the rest of the statement

Consider the proposition, $\exists x\ (x>5 \wedge x<2)$.
This asserts that there exists a real number which is both greater than 5 and less than 2. We suspect that this statement is false. So its negation would be true. What is its negation?
$\neg\ [\ \exists x\ (x>5 \wedge x<2)\ ] \equiv \forall x\ \neg\ (x>5 \wedge x<2) \equiv \forall x\ (x\leq 5 \vee x\geq 2)$.

Note that the negation of the existential statement is a universal statement, so to prove that the existential statement is false, we can show that the universal statement is true and for this we need a general argument.

Prove $\forall x\ (x\leq 5 \vee x\geq 2)$.
Proof: Let x be any real number.
Now, since x is a real number, x is either ≤ 5 or >5.
If $x\leq 5$ then the sentence $x\leq 5 \vee x\geq 2$ is true because an "or" statement is true if one or the other or both are true (definition of "or").
If $x>5$ then certainly $x\geq 2$ so, again, the sentence $x\leq 5 \vee x\geq 2$ is true because an "or" statement is true if one or the other or both are true (definition of "or").
Since, in either case (if $x\leq 5$ or if $x >5$), the sentence $x\leq 5 \vee x\geq 2$ is true, therefore it is always true. Thus we have shown that $\forall x\ (x\leq 5 \vee x\geq 2)$ is true.

By the way, what valid argument are we using here?
What is the structure of this argument?
Either $x \leq 5$ or $x > 5$.
If $x \leq 5$ then $x \leq 5 \vee x \geq 2$.
If $x > 5$ then $x \leq 5 \vee x \geq 2$.

$x \leq 5 \vee x \geq 2$ or $x \leq 5 \vee x \geq 2$ by the valid argument.......
So, $x \leq 5 \vee x \geq 2$ because.....

On the other hand to negate a universally quantified proposition we change the quantifier to an existential quantifier and then negate the rest of the statement.

To negate the statement: $\forall x \ (x^2 > 9 \rightarrow x + 7 > 10)$.

we change the quantifier to an existential quantifier and then negate the rest of the statement. $\exists x \ \neg (x^2 > 9 \rightarrow x + 7 > 10)$.
Now, how do we negate the rest of the proposition?
What kind of proposition is it?

Thus we get: $\exists x \ (x^2 > 9 \wedge x + 7 \leq 10)$.
Is this true? Can you find an x which satisfies this proposition?
Therefore the original proposition, $\forall x \ (x^2 > 9 \rightarrow x + 7 > 10)$, is false.

Here are some problems for **practice:**
First find the negation of each proposition.
Then determine whether the original proposition is true (in which case its negation is false) or whether the original proposition is false (in which case the negation is true). Assume that we are working in the real numbers unless otherwise indicated. For the false propositions give a counter example. Write a proof of the true ones.

1) $\forall x \ (x \le 12)$
2) $\exists x \ni (x > 30)$
3) $\forall x \exists y \ni (x - y) > 3$
4) $\exists x \forall y \ (x - y) > 3$
5) $\forall x \forall y \ (x - y) > 3$
6) $\exists y \exists x \ni (x - y) > 3$
7) $\forall x \ (x^2 \ge 0 \ \wedge \ x < 2)$
8) $\forall x \ (x^2 \ge 0 \ \vee \ x < 2)$
9) $\forall x \ (x < 2 \ \rightarrow \ x^2 \ge 0)$
10) $\exists z \ni (z < 5 \ \vee \ z \ge 7)$
11) $\exists z \ni (z < 5 \ \vee \ z \ge 1)$
12) $\exists z \ni (z < 5 \ \rightarrow \ z \ge 7)$
13) $\exists x \ni \forall y \ (xy = y)$
14) $\exists x \ni \forall y \ (xy = x)$
15) $\forall x \ \exists y \ (xy = y)$
16) $\forall x \ \exists y \ (xy = x)$
17) $\forall x \ \exists y \ (xy = 5)$
18) $\forall x \ \exists y \ (y=0 \ \vee \ xy = 5)$
19) $\exists y \ (y < 7 \ \vee \ y \ge 2)$
20) $\exists x \ni \forall y \ ((x > y) \ \rightarrow \ (x > 0))$
21) $\forall x \ \exists y \ni ((x > y) \ \rightarrow \ (x > 0))$
22) $\forall y \ \exists x \ni ((x > y) \ \rightarrow \ (x > 0))$

Exercises: First find the negation of each proposition.
Then determine whether the original proposition is true (in which case its negation is false) or whether the original proposition is false (in which case the negation is true). Assume that we are working in the real numbers unless otherwise indicated. For the false propositions give a counter example. Write a proof of the true ones.
Determine which of the following are true and which are false. Assume that we are working in the real numbers unless otherwise indicated. For the false ones give a counter example. Write a proof of the true ones.

1) $\forall x \ (x > 103)$
2) $\forall x \ \exists y \ni (x + y = 17)$
3) $\exists x \ni \forall y \ (x + y = 17)$
4) $\exists y \ni \forall x (x^2 = y)$
5) $\forall x \exists y \ni (x^2 = y)$
6) $\forall x \ \exists y \in$ rational number $\ni \sqrt{x} = y$
7) $\forall x \exists y \ni \sqrt{x} = y$
8) $\exists x \forall y \ \ x+y = 10$
9) $\forall x \exists y \ni \ x+y = 10$
10) $\forall x \ \exists y \ni \ y = \frac{1}{x}$
11) $\forall x \ \forall y \ \ x < y$
12) $\forall x \ \exists y \ \ x < y$
13) $\exists x \ \forall y \ \ x < y$
14) $\forall x \ \exists y \ \forall z \ \ x + y < z$
15) $\exists x \ni \forall y \ \exists z \ni ((x>y) \rightarrow (y>z))$
16) $\forall x \in$ natural numbers $\exists y \in$ natural numbers $\ni xy = x$
17) $\exists y \in$ natural numbers $\forall x \in$ natural numbers $\ni xy = x$

18) $\forall x \in$ real numbers $\exists y \in$ real numbers $\ni x + y = 0$
19) $\exists x \in$ real numbers $\forall y \in$ real numbers $\ni x + y = 0$
20) $\exists x \ni \forall y \ \exists z \ni (x+y = z) \vee (x > y-z))$

Classical logic talks about categorical propositions. These are propositions that make assertions affirming or denying that one class (or set or category) of objects is included, either entirely or in part, in another class (or set or category) of objects.

Here are 4 types of classical categorical propositions.
The first type we might call universal affirmation.
All X are Y Examples: All cats smile. All lawyers are liars.

They are various ways to write an All X are Y proposition:
Every cat smiles. Every lawyer is a liar.
Each cat smiles. Each lawyer lies.
Any cat smiles. Any lawyer lies.

Obviously you can see how the ambiguity and subtlety of a natural language can add or subtract meaning to such categorical statements.
The only intent is All X are Y. Everything that is in the class or set X is also in the class or set Y.

The second type we might call universal negation.
All X are not Y or **No X is Y**
Examples: All cats do not smile. All lawyers are not liars.
All trees are not rocks. All desks are not cars.
There is some real ambiguity in here.
What we mean is: Each and every object or element in the set X does not belong to the set Y. No object in X belongs to Y
No cat smiles. No lawyer is a liar. Every desk is not a car.
No tree is a rock.

The third type we might call existential affirmation.
Some X are Y or **There exists an X which is Y**
Examples: Some people are jejune. There exists a green lion.
Some numbers are prime.

This asserts the existence of one object in X which has the property Y. Possibly one or possibly more than one, but at least one of the objects in X belongs to Y.

The fourth type we might call existential negation.
Some X are not Y or **There exists an X which is not Y**
Examples: Some gazelles are not graceful.
There is a poem that does not rhyme. Not all sheep have wool.
Some numbers are not prime.

This asserts the existence of one object in X which does not have the property Y. Possibly one or possibly more than one, but at least one of the objects in X does not belongs to Y.

How do we negate such statements?

For the statement All X are Y
the negation is Some X are not Y or There exists an X which is not Y

For the statement All X are not Y or No X is Y
the negation is Some X are Y or There exists an X which is Y

For the statement Some X are Y or There exists an X which is Y
the negation is All X are not Y or No X is Y

For the statement Some X are not Y or There exists an X which is not Y
the negation is All X are Y

So, the negation of the universal statement
All birds can fly.
is
There is a bird that can not fly
or, There exists at least one bird that cannot fly.

The negation of the universal statement
All statues are not made of yogurt.
That is, No statue is made of yogurt.
is There is a statue made of yogurt
or, There exists at least one statue that is made of yogurt.

The negation of the existential statement
Some trees are musicians.
is
All trees are not musicians
or, No tree is a musician.

The negation of the existential statement
Some cabins are not rustic.
is
All cabins are rustic.
or, No cabin is not rustic.

Thus the negation of a universal statement is an existential statement where we negate the rest of the statement, and the negation of an existential statement is a universal statement where we negate the rest of the statement.

Exercises: For each statement below, write it in symbolic form, then write its negation symbolically.
Determine whether the statement or its negation is true.

21) For every real number x there is a real number y such that xy = 17.
22) There exists a natural number n such that for every natural number m n plus m is equal to n squared.
23) For every real number x there is a natural number n such that $1/n < x$
24) For every positive real number x there is a natural number n such that $1/n < x$
25) There is no largest rational number.
26) If a proposition has n different statements in it, then the truth table for that proposition has 2^n rows.
27) If P and $P \to Q$ are both propositions in an argument then Q follows logically.
28) If Q and $\neg P \vee (\neg Q)$ are both propositions in an argument then $\neg P$ follows logically.

We often give definitions of mathematical objects or concepts in terms of quantifiers.

A function f is <u>increasing</u> iff
$\forall x \; \forall y \quad x \le y \;\; \to \;\; f(x) \le f(y)$

A function f is <u>continuous</u> at a point c iff
for every $\forall \varepsilon > 0 \; \exists \delta > 0 \;\; \ni \;\; |x - c| < \delta \;\; \to \;\; |f(x) - f(c)| < \varepsilon$

L is the <u>limit</u> of a function f at a point c iff
for any $\varepsilon > 0$ there is a $\delta > 0 \;\; \ni \;\; 0 < |x - c| < \delta \;\; \to \;\; |f(x) - L| < \varepsilon$
<u>Write</u> the negation of each of the above three definitions.

Intermediate Value Theorem: If f is a continuous function on the closed interval [a, b], and m is between f(a) and f(b) then there is a point c in the closed interval [a, b] such that f(c) = m.

If we look at it carefully we can even see that the statement has an implied quantifier for the function. Namely, FOR EVERY FUNCTION f, …

Rolle's Theorem: If f is a function which is continuous on the closed interval [a, b], differentiable on the open interval (a, b) and f(a) = f(b) then there is at least one number c in the open interval (a, b) such that f ' (c) =0.

Exercise: Write each of the two previous theorems in symbolic form.
Then write the negation symbolically.
Finally write the negation in English.

Practice problem: some solutions:
1) False. Counterexample: Let $x = 15$
2) True. Let $x = 31$
3) True. Proof: Let x be any real number. Let $y = x - 4$.
Then $x - y = x - (x - 4) = 4$, so $x - y$ is always > 3. QED

4) False. We take the negation and prove that it is true.
The negation is
For every x there exists a y such that $x - y$ is NOT >3.
So,
For every x there exists a y such that $x - y$ is less than or equal to 3.
Proof: Let x be any real number. Let $y = x + 1$.
Then $x - y = x - (x + 1) = -1$, so $x - y$ is always < 3.
Since the negation is true then the original statement is false.

5) False. Counterexample: Let $x = 2$ and let $y = 1$.
Then $x - y = 2 - 1 = 1$, which is < 3.
6) True. Let $x = 7$ and let $y = 3$, then $x - y = 7 - 3 = 4$, which is > 4. QED
7) False. Counterexample: let $x = 5$, so $x<2$ is false so the "and " statement is false.
Thu s it is not true for ALL x.

8) True. Since the first part of the "or" statement is always true, then the "or" statement is always true (for all x).
9) True. Here the second part of the implication is always true thus the statement is always true (for all x) therefore the universally quantified statement is true.
10) True. Let $z = 4$ so the "or" statement is true.
11) similar to 10)
12) True in the dumb way. Make the first part of the implication false, let $z = 9$.
13) True. Let $x = 1$, then for any real number y, $xy = y$ (since $x = 1$).
(note that $xy = y$ means $xy - y = 0$, so $y(x - 1)=0$ and no other real number for y will work FOR ALL y.)
14) True. Let $x = 0$, then for any real number y, $xy = x$ (since $x = 0$).
(note that $xy = x$ means $xy - x = 0$, so $x(y - 1)=0$ and no other real number for x will work FOR ALL Y.)
15) True. Let x be any real number. Let $y = 0$.

Note that 14) is a stronger statement because it says there exists a single number that works for all real numbers, while in 15), for each real number we can find a number that works for just that number.

16) True. Let x be any real number. Let y = 1.
Note that 13) is a stronger statement because it says there exists a single number that works for all real numbers, while in 16), for each real number we can find a number that works for just that number.

17) False. Counterexample: Let $x = 0$ then no matter what y we pick xy will not $= 5$.
Observe that our counterexample is the negation of the statement.
The negation is: There exists an x such that for every y, xy is not = 5.
18) True. In fact something stronger is true.
There exists a y such that for any x, $y = 0$ or $xy = 5$. (Let $y = 0$)
19) True. Let $y = 6$.
20) True. Let $x = 2$, so $x>0$ is true, so the implication is true for any y and no matter what $x>y$ is.
21) True. Let x be any real number then let y be less than x, so $x>y$ is false, so the implication is true.
22) True. Let y be any real number then let x be less than y, so $x>y$ is false, so the implication is true.

Exercises: some solutions
1) False.
2) True. Let x be any real number and then let $y = (17 - x)$.
So $x + y = x + (17 - x) = 17$
3) False. The negation of the statement is:
For every x there exists a y such that x + y is NOT = to 17.
Proof of negation: Let x be any real number and then let $y = (3 - x)$.
Thus $x + y = x + (3 - x) = 3$ which is NOT = 17.

4) False. The negation of the statement is:
For every y there exists a x such that x^2 is NOT = to y.
Proof of negation: Let y be any real number and then, if $y=0$ or $y>0$, let $x = \sqrt{y-1}$. Thus, $x^2 = (\sqrt{y-1})^2 = y - 1$, which is NOT = y.
If $y < 0$ then let $x = 1$, so x^2 will be $=1$ which is not < 0.
In any case, x^2 will not = y.

5) True. Let x be any real number and let $y = x^2$ (since, given any real number we CAN square it). So $x^2 = y$.

6) False. Counterexample: Let $x = 7$, then there is no real number y such that $y = \sqrt{7}$.
That is, there exists an x such that for any y, $\sqrt{x}$ is NOT = y.

7) False. Counterexample: Let $x = -5$. There is no real number y such that $y = \sqrt{-5}$ (since $\sqrt{-5}$ is not a real number).
That is, there exists an x such that for any y, $\sqrt{x}$ is NOT = y.

8) False. The negation of the statement is:
For every x there exists a y such that x + y NOT = to 10.
Proof of negation: Let x be any real number and then, let $y = 2 - x$.
Thus, $x + y = x + (2 - x) = 2$ which is NOT = 10.

9) True. Let x be any real number and let $y = 10 - x$.
Thus, $x + y = x + (10 - x) = 10$ which is = 10.

10) False. Let x =0….
11) False. let x = 2 and let y = 1.

13) False. The negation of the statement is:
For every x there exists a y such that x is NOT < y.
Proof of negation: Let x be any real number and then, let $y = x - 1$.
Thus, $y = x + 1$ so x is NOT < y.

14) False. The negation is:
There exists an x such, that for every y, there exists a z such that x + y is greater than or equal to z.
Let x = 0, then let y be any real number, now let $z = y - 1$. This means that $y = z + 1$.
So $x + y = 0 + (z + 1) = z + 1$ and $z+1$ is > z. Thus x + y is greater than or equal to z.
15) True.
Proof: Let x = 0. Then let y be any real number. Now let $z = y - 4$.
$z = y - 4$, so $z+4 = y$ which means that $y > z$. Thus the consequent of the implication is true, so the whole implication is true.

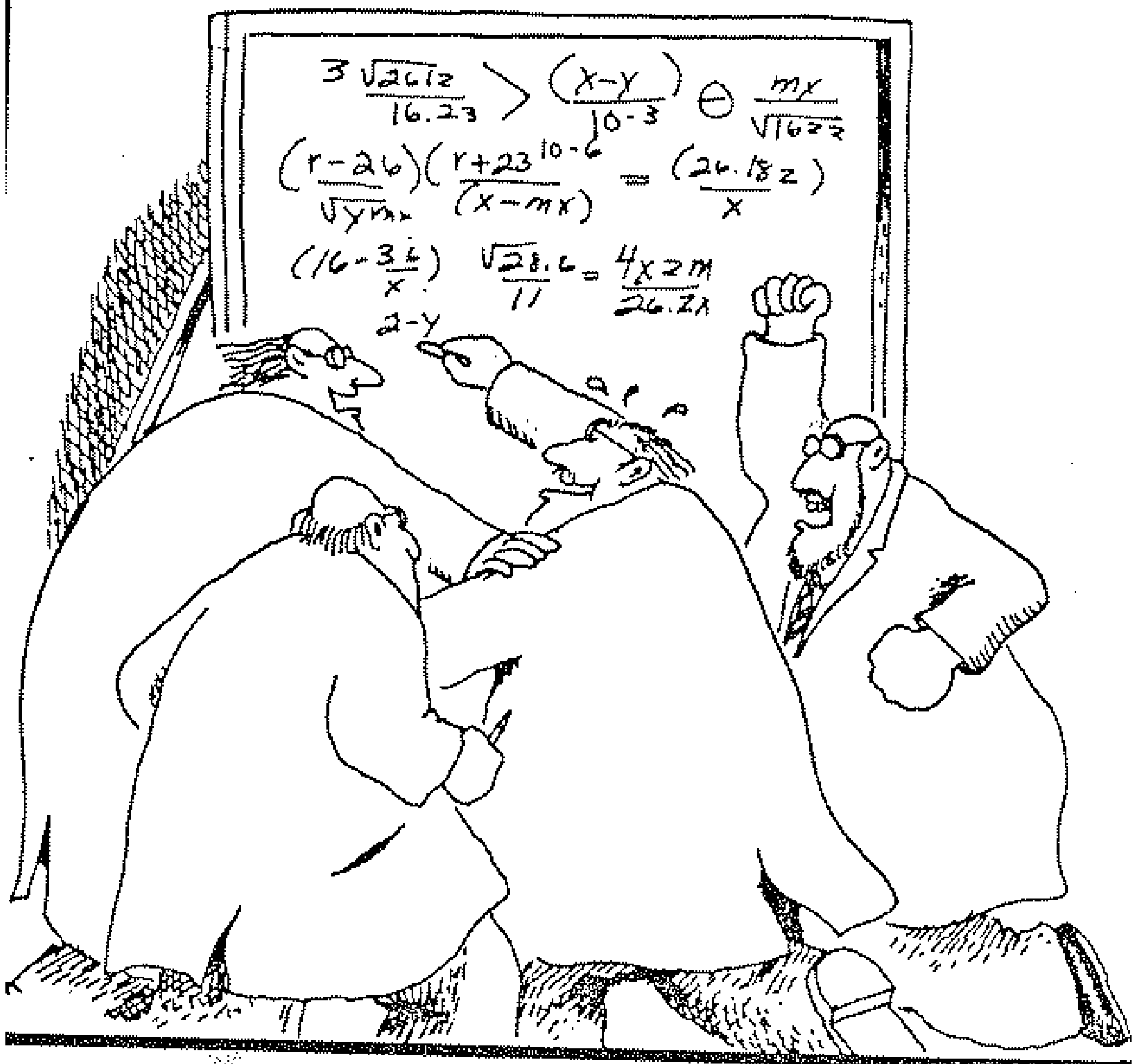
16) True. This is like 16) on the previous page except now we are saying that the numbers must be natural numbers, 1,2,3,4,5... So, let y=1.

17) True. This is like 13) on the previous page except now we are saying that the numbers must be natural numbers, 1,2,3,4,5... So, let y=1.

18) True. Proof: Let x be any real number and then let $y = -x$.
Thus $x + y = x + (-x) = 0$.
19) False. No x will work for every y. The negation of the statement is:
For every real number x there exists a real number y such that x + y is NOT = 0.
Proof of negation: Let x be any real number and then, let $y = 1 - x$.
Thus, $x + y = x + (1 - x) = 1$, and 1 is NOT = 0.

We have completed the first part of the course. We have discussed propositional logic, valid arguments and quantifiers. There is much to learn here and it is essential to what we will do next in this course and what you will do in your future mathematical studies. Make sure you know it completely and well. Make this information part of your knowledge and understanding. This will probably require doing more, and working more than you are used to. This is a time to upgrade your skills, habits and abilities.

When one studies mathematics long enough one eventually reaches a place where native intelligence and quickness of grasp (no matter how extensive, powerful and impressive) no longer carries one through the details. At this point determination <u>and</u> <u>consistent application of hard work</u> on the material create a giant step in the growth of knowledge, in technique, and in what is called mathematical maturity. This growth cannot occur if one relies entirely on native intelligence. We must put in the daily time and energy needed to learn definitions, theorems, proofs and toil through the exercises and examples. Make sure you read through this material carefully. You may need to read through it several times. Keep engaging it. Ask questions. The more you work on these concepts the simpler they will seem. You can do this. Work hard and you will enjoy the pleasure and celebrate the reward of understanding.



"Go for it, Sidney! You've got it! You've got it! Good hands! Don't choke!"